Differential and Integral Calculus.

Introduction

In this session we review differential and integral calculus, concentrating for the time being on functions of a single real variable; these restrictions will be relaxed next time. We also have a little bit to say about differential equations; as we limit ourselves to material that is directly relevant to radar performance we will hardly scratch the surface of this subject. A lot of today's session will be review of material that is already familiar; this has the advantage of easing us gradually into new territory. Unfortunately, unfamiliar mathematical technique can only be mastered through exposure, explanation and practice, so there will be rather more worked examples this time. These should also allow us to review a lot of the background material that was glossed over in the panoramic view of things presented in the previous session. There is a set of exercises provided with these notes. Some of these are straightforward and will help you build up your confidence in formal manipulations (it helps that they deal with relatively familiar material); the occasional complete (but more interesting) bastard also lurks amongst them.

Differential Calculus

Rates of change: chords and tangents

Slope of chord between $(x, y(x))$ and $(x+\Delta x, y(x+\Delta x))$

$$\frac{\Delta y}{\Delta x} = \frac{y(x+\Delta x) - y(x)}{\Delta x}$$

As Δx gets smaller this ratio tends to the slope of the tangent of the curve $y = y(x)$ at the point $(x, y(x))$. This we identify as the rate of change of the function $y(x)$, which we refer to as its first derivative. The notation we use is derived from our cruder expression for the slope:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

We will do nothing further to justify this procedure, but will set about working out the rules of thumb required to calculate the derivative of a given function.

Simplest example

$$'y(x)' = C$$
$$\frac{\Delta y}{\Delta x} = \frac{0}{\Delta x}; \quad \frac{dy}{dx} = 0$$

The condition $dy/dx = 0$ is characteristic of a 'stationary point' where the tangent to the curve is parallel to the x axis(i.e. the rate of change of the function is zero)

Slightly harder example

$$y(x) = ax + b$$
$$\frac{\Delta y}{\Delta x} = \frac{a\Delta x}{\Delta x}; \quad \frac{dy}{dx} = a$$

The identification of the derivative with the slope is borne out in this special case of the straight line. (it's always a good thing to check on special cases.)

Powers

$$\begin{aligned}
&y(x) = x^2 \\
&\Delta y = (x+\Delta x)^2 - x^2 = 2x\Delta x + (\Delta x)^2 \\
&\frac{\Delta y}{\Delta x} = \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = 2x + \Delta x \\
&\frac{dy}{dx} = 2x
\end{aligned}$$

Using the binomial theorem (really we only need the first two terms as the rest go to nothing in the limiting process) we can establish that

$$\begin{aligned}
&y(x) = x^\alpha \\
&\frac{dy}{dx} = \alpha x^{\alpha-1}
\end{aligned}$$

Now that we can differentiate individual powers of *x* we can differentiate power series

$$\begin{aligned}
&y = \sum_{r=0}^{\infty} a_n x^{n+\alpha} \\
&\frac{dy}{dx} = \sum_{r=0}^{\infty} (n+\alpha) a_n x^{n+\alpha-1}
\end{aligned}$$

A particularly important example of this is the exponential function

$$\begin{aligned}
&y(x) = \exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
&\frac{dy}{dx} = \sum_{n=0}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = y
\end{aligned}$$

Trigonometric functions

$$\begin{aligned}
&y(x) = \sin(x) \\
&y(x+\Delta x) = \sin(x)\cos(\Delta x) + \cos(x)\sin(\Delta x) \\
&\qquad\qquad \approx \sin(x) + \cos(x)\Delta x \\
&\frac{\Delta y}{\Delta x} \approx \cos(x); \quad \frac{dy}{dx} = \cos(x) \\
&y(x) = \cos(x); \quad \frac{dy}{dx} = -\sin(x)
\end{aligned}$$

More general results can be derived in much the same way, once we make the series expansion (a bit of a Taylor series)

$$f(x+\Delta x) \approx f(x) + \frac{df(x)}{dx}\Delta x$$

Thus

$$y(x) = f(x)g(x)$$
$$y(x+\Delta x) = f(x+\Delta x)g(x+\Delta x) \approx f(x)g(x) + (f(x)g'(x) + g(x)f'(x))\Delta x$$
$$\frac{\Delta y}{\Delta x} \approx \frac{(f(x)g'(x) + g(x)f'(x))\Delta x}{\Delta x}; \quad \frac{dy}{dx} = f\frac{dg}{dx} + g\frac{df}{dx}$$

$$y(x) = f(x)/g(x)$$
$$y(x+\Delta x) = f(x+\Delta x)/g(x+\Delta x) \approx f(x)/g(x) + \left(f'(x)/g(x) - f(x)g'(x)/g(x)^2\right)\Delta x$$
$$\frac{\Delta y}{\Delta x} \approx \frac{\left(f'(x)/g(x) - f(x)g'(x)/g(x)^2\right)\Delta x}{\Delta x} \quad \frac{dy}{dx} = \frac{f'g - g'f}{g^2}$$

The ‘chain rule’

$$y(x) = f(g(x))$$
$$y(x+\Delta x) = f(g(x+\Delta x)) \approx f(g(x) + g'(x)\Delta x) \approx f(g(x)) + \frac{df}{dg}\frac{dg}{dx}\Delta x$$
$$\frac{\Delta y}{\Delta x} \approx \frac{\frac{df}{dg}\frac{dg}{dx}\Delta x}{\Delta x}; \quad \frac{dy}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

Inverse functions

$$y = f^{-1}(x); \quad x = f(y)$$
$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{f'(y)} = \frac{1}{f'\left(f^{-1}(x)\right)}$$

A few examples may make the use of these rules more clear.

The tangent is built up from the sine and the cosine; thus its first derivative can be built up from those of its constituent parts as follows

$$y(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$$
$$\frac{dy}{dx} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \sec^2(x) = 1 + \tan^2(x)$$

An example of differentiating a function of a function:

$$y(x) = \exp\left(-ax^2\right): \quad 'f' \equiv \exp, 'g' \equiv -ax^2$$
$$\frac{df}{dg} = \exp(g); \quad \frac{dg}{dx} = -2ax$$
$$\frac{dy}{dx} = -2ax\exp\left(-ax^2\right)$$

The logarithm is the inverse function of the exponential

$$y(x) = \log(x); \quad x = \exp(y)$$
$$\frac{dx}{dy} = \exp(y); \quad \frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{\exp(y)} = \frac{1}{x}$$
$$\frac{d\log(x)}{dx} = \frac{1}{x}$$

Another example of differentiating an inverse function

$$y(x) = \tan^{-1}(x); \quad x = \tan(y)$$
$$\frac{dx}{dy} = 1 + \tan^2(y); \quad \frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{1+\tan^2(y)} = \frac{1}{1+x^2}$$

There are plenty more examples in the exercises. Differentiation is basically quite straightforward and systematic; given any function you can hack out its derivatives.

Repeated differentiation

$$\frac{d^n y}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)$$

Leibnitz rule for the repeated differentiation of a product of two functions

$$\frac{d^n}{dx^n}\left(f(x)g(x)\right) = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} \frac{d^r f(x)}{dx^r} \frac{d^{n-r} g(x)}{dx^{n-r}}$$

Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{\left(x - x_0\right)^n}{n!} \frac{d^n f\left(x_0\right)}{dx_0^n}$$

$$\left.\frac{d^n}{dx^n}\left(x - x_0\right)^m\right|_{x=x_0} = n!\,\delta_{n,m}$$

A neat example (there's one very like it in the exercises if you fancy it.)

If $y(x) = \tan^{-1}(x)$ show that

$$\left(1+x^2\right)y_{n+1}+2nxy_n+n(n-1)y_{n-1}=0;\quad y_n\equiv\frac{d^n y}{dx^n}$$

and deduce the Taylor series expansion around x=0.

.
<u>Integral Calculus</u>

While differential calculus allows us to characterise small (infinitesimal) changes in a function; integral calculus provides us with the formal machinery with which to sum up and combine (integrate) many such successive small changes to recover their resultant whole. A simple graphical intuition identifies such an integral with an area under a curve. Integration can also be viewed as the inverse of differentiation. Just as we were able to establish a set of rules for differentiating the familiar elementary functions, so we can 'run these in reverse' to obtain the corresponding set of rules for integration. The rules for differentiation are straightforward and allow us to differentiate any well behaved function, essentially without thought. Integration, being an inverse problem, is much more tricky, so much so that many quite simple looking functions cannot be integrated in terms of the elementary functions. We can get round this problem in several ways: transforming and approximating the integral in limits of physical interest, identifying 'special functions' that effect the integrations for us and defining functions in terms of frequently occurring but intractable integrations. As the integrals that we meet in practical problems are not (unlike textbook and examination questions) chosen to be 'doable' in closed form through some clever trick we will spend rather more time on these techniques than on the more conventional 'recipes for integration' approach, which can always be accessed through Mathematica or tabulated results..

Definite integration

$$S_N=\sum_{j=0}^{N}f\left(a+j\Delta_N\right)\Delta_N;\quad \Delta_N=\frac{\left(b-a\right)}{N}$$

$$\int_a^b f\left(x\right)dx=\lim_{N\to\infty}S_N$$

Geometrical intuition: area under the curve $y=f\left(x\right)$ between $x=a$ and $x=b$

Indefinite integration

$$F\left(x\right)=\int_a^x f\left(x'\right)dx'\quad \frac{dF\left(x\right)}{dx}=f\left(x\right)\quad \int_a^b f\left(x'\right)dx'=F\left(b\right)-F\left(a\right)$$

Rules for indefinite integration (c.f. rules for differentiation)

$$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1}$$
$$\int \sin(x)dx = -\cos(x)$$
$$\int \cos(x)dx = \sin(x)$$
$$\int \frac{dx}{x} = \log(x)$$
$$\int \exp(x)dx = \exp(x)$$
$$\int \frac{dx}{1+x^2} = \tan^{-1}(x)$$

The rules for differentiating a function of a function and a product of two functions furnish us with two very useful tricks: substitution and integration by parts. This is where experience and low cunning start to come into play e.g.

$$\int x\exp\left(-ax^2\right)dx = \frac{1}{-2a}\int \exp\left(-ax^2\right)\frac{d\left(-ax^2\right)}{dx}dx = \frac{1}{-2a}\int \exp(y)dy; \quad y = -ax^2$$
$$= \frac{1}{-2a}\exp\left(-ax^2\right) + C$$
$$\int \tan(x)dx = \int \frac{\sin(x)}{\cos(x)}dx = -\int \frac{1}{\cos(x)}\frac{d\cos(x)}{dx}dx = -\int \frac{1}{y}dy; \quad y = \cos(x)$$
$$= -\log(\cos(x)) + C$$

An A level example

$$\int \frac{x+1}{a^2+x^2}dx = \int \frac{x}{a^2+x^2}dx + \int \frac{1}{a^2+x^2}dx$$
$$= \frac{1}{2}\log\left(a^2+x^2\right) + \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Integration by parts is based on the expression for the derivative of a product of two functions

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating this relation with respect to x gives us

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

which is commonly re-written in the form

$$\int f(x)g(x)dx = f(x)\int^x g(x')dx' - \int\left(\int^x g(x')dx'\right)\frac{df(x)}{dx}dx\ .$$

Here are a couple of examples:

$$\int \log(x)dx = x\log(x) - \int x\frac{d}{dx}\log(x)dx = x\log(x) - \int dx = x\log(x) - x + C$$

$$\int \sin^{-1}(x)dx = x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}dx = x\sin^{-1}(x) + \sqrt{1-x^2} + C$$

Definite integrals and principal values.

When an indefinite integral can be worked out explicitly it is readily converted into a definite integral by substituting the limits of integration into the answer. This usually doesn't give rise to too much trouble, unless the indefinite integral takes infinite values at one or both of the limits. Thus we might have:

$$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1}(x)\Big|_0^1 = \frac{\pi}{4}$$

$$\int_0^\infty \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\Big|_0^\infty = \frac{\pi}{2a}$$

$$\int_1^\infty \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^\infty = 1$$

$$\int_0^\infty \frac{dx}{x^2} \quad \text{'non-integrable singularity at the origin'}$$

You might also come across a singularity in the integrand inside the range of integration. Let's say the range is (a,b) and there is a singularity in the integrand $f(x)$ at $x = c, a < c < b$. The integral is then evaluated as

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)dx + \lim_{\eta \to 0} \int_{c+\eta}^b f(x)dx$$

The Cauchy principal value of the integral is defined by

$$P\int_a^b f(x)dx = \lim_{\varepsilon \to 0}\left(\int_a^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^b f(x)dx\right)$$

If the singularity at c is integrable the Cauchy principal value coincides with conventional value of the integral; if the singularity goes off to infinity like an odd function of (x-c) then the PV can exist, even though the conventional interpretation of the integral is not well defined. e.g.

$$P\int_{-\infty}^{\infty} \frac{\exp(ix)}{x}dx = i\int_{-\infty}^{\infty} \frac{\sin(x)}{x}dx$$

which is finite (its evaluation is an exercise).

So far, so good. This is the sort of stuff you do at school/College and, while it might have been a bit rusty half an hour ago, is now doubtless flooding back. There are, however, two techniques that seem to get passed over in most mathematical methods courses that are, nonetheless, very useful. These involve differentiating and integrating under the integral sign. The validity of these procedures is a matter of some delicacy, which we will, in line with our usual practise, overlook here.

Differentiating under the integral sign.

If the integrand, and consequently the resulting integral, depend on some parameter then differentiating each with respect to that parameter gives the same answer. i.e. if

$$F(\alpha) = \int_a^b f(x,\alpha)dx$$

then

$$\frac{\partial F(\alpha)}{\partial \alpha} = \int_a^b \frac{\partial f(x,\alpha)}{\partial \alpha}dx$$

Frequently it is easier to recognise F and form $\frac{\partial F(\alpha)}{\partial \alpha}$ than it is to evaluate the corresponding integral over x. A couple of examples should make things clearer:

$$\int_0^\infty dx \exp(-ax) = \frac{1}{a} \equiv F(a)$$

$$\frac{dF}{da} = -\frac{1}{a^2} = -\int_0^\infty dx x \exp(-ax)$$

$$\frac{d^2F}{da^2} = \frac{2}{a^3} = \int_0^\infty dx x^2 \exp(-ax)$$

$$\frac{d^nF}{da^n} = \frac{n!}{a^{n+1}} = \int_0^\infty dx x^n \exp(-ax)$$

This saves us an awful lot of integrating by parts (check it if you like)

$$\int_0^\infty \exp(-x)\cos(bx)dx = \frac{1}{\left(1+b^2\right)}$$

$$\int_0^\infty x\exp(-x)\sin(bx)dx = -\frac{d}{db}\frac{1}{\left(1+b^2\right)} = \frac{2b}{\left(1+b^2\right)^2}$$

Differentiating under the integral sign turns out to be good way of evaluating integrals containing logarithms; we have to recall that

$$\frac{d}{d\alpha}x^{\alpha} = \frac{d}{d\alpha}\exp(\alpha\log(x)) = \log(x)\exp(\alpha\log(x)) = \log(x)x^{\alpha}$$

Example 1 contains a couple of exercises that are best tackled in this way.

Reversing the order of integration.

Sometimes it happens that part of the integrand, which is a bit nasty to handle as it stands, can be represented as an integral of a more tractable function. If this ‘integral representation’ is plugged into the original integral and the order of integration is reversed things frequently sort themselves out much better. Once again a few examples should make things clearer.

$$\int_0^\infty \frac{\exp(-ax)-\exp(-bx)}{x}dx = \int_0^\infty\left(\int_a^b \exp(-sx)ds\right)dx = \int_a^b\left(\int_0^\infty \exp(-sx)dx\right)ds = \int_a^b \frac{ds}{s} = \log(b/a)$$

$$\begin{aligned}\int_0^\infty \frac{t^n}{(1+t)^{m+n+1}}dt &= \frac{1}{(m+n)!}\int_0^\infty t^n\left(\int_0^\infty \exp(-s(1+t))s^{m+n}ds\right)dt \\ &= \frac{1}{(m+n)!}\int_0^\infty \exp(-s)s^{m+n}\left(\int_0^\infty t^n\exp(-st)dt\right)ds \\ &= \frac{n!}{(m+n)!}\int_0^\infty \exp(-s)s^{m-1} \\ &= \frac{n!(m-1)!}{(m+n)!}\end{aligned}$$

$$\int_0^\infty \frac{\sin(x)}{x}dx = \int_0^\infty \sin(x)\left(\int_0^\infty \exp(-tx)dt\right)dx = \int_0^\infty\left(\int_0^\infty \exp(-tx)\sin(x)dx\right)dt = \int_0^\infty \frac{dt}{1+t^2} = \frac{\pi}{2}$$

The exercises contain several examples that are best attacked by this approach.

Asymptotic approximations to integrals

Expansions of integrals involving a large parameter are often very useful. The basic idea is to locate the range in which the integrand makes a non-negligible contribution to the integral, then construct a suitable approximation to this. The integrand makes a negligible contribution to the integral if either its value is very small (as a result of exponential decay) or is oscillating so furiously that it cancels itself out. We will consider these two mechanisms of suppression separately.

Rapid decay from the lower bound of integration.

e.g. the complementary error function

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \exp\left(-t^2\right) dt = \frac{1}{\sqrt{\pi}} \int_{z^2}^\infty \frac{\exp(-p)}{\sqrt{p}} dp$$

Successive integrations by parts generate a series that captures the large z behaviour quite effectively.

$$\frac{1}{\sqrt{\pi}} \int_{z^2}^\infty \frac{\exp(-p)}{\sqrt{p}} dp = \frac{1}{\sqrt{\pi}} \left(\frac{\exp\left(-z^2\right)}{z} - \frac{1}{2} \int_{z^2}^\infty \frac{\exp(-p)}{p^{3/2}} dp \right)$$

$$\sim \frac{\exp\left(-z^2\right)}{z\sqrt{\pi}} \left(1 + \sum_{n=1}^\infty (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{\left(2z^2\right)^n} \right)$$

Series of this type are described as asymptotic; they tend to converge rapidly to a good approximation to the 'right' answer after a few terms then, as they are generally divergent, rattle off to an infinite and meaningless answer as more terms are included. For many purposes just the first term is good enough; controlling and interpreting high order terms in asymptotic expansions is a subject of active research.

The integration by parts technique is very useful. An alternative approach exploits Watson's lemma. Basically, if you can write your integral in the form

$$F(\gamma) = \int_0^\infty \exp(-\gamma t)\, f(t) dt$$

and f can be expanded in a power series with a non-vanishing radius of convergence

$$f(t) = t^\alpha \sum_{n=0}^\infty f_n t^n; \quad |t| < \delta$$

then

$$F(\gamma) \sim \int_0^\infty dt \exp(-\gamma t) t^\alpha \sum_{n=0}^\infty f_n t^n = \sum_{n=0}^\infty f_n \frac{\Gamma(n+\alpha+1)}{\gamma^{n+\alpha+1}}; \gamma \to \infty .$$

Crudely speaking, the rapidly decaying exponential has killed off the power series before it has had chance to get outside its radius of convergence and an asymptotic series has been derived. You might like to derive the expansion of the complementary error function using Watson's lemma.

Rapid decay around a point within the range of integration

We now consider integrals of the form

$$F(\gamma) = \int_a^b \exp(\gamma g(x)) h(x) dx; \quad \gamma \to \infty$$

where g takes a maximum value at $x = x_0$ within the range of integration. h is a slowly varying function of x. Because the exponential term decays very rapidly away from $x = x_0$ we expand the exponent about this point; as g has a stationary point at $x = x_0$ the decay away from this point will be like a Gaussian (usually, at least). Thus we write

$$\left.\frac{dg(x)}{dx}\right|_{x=x_0} = 0 \quad \text{(stationary point)}$$

$$g(x_0 + p) \approx g(x_0) - \frac{\Lambda}{2}p^2; \quad \Lambda = -\left.\frac{d^2g(x)}{dx^2}\right|_{x=x_0} > 0 \ (g \text{ at maximum})$$

$$\begin{aligned} F(\gamma) &\sim h(x_0)\exp(\gamma g(x_0))\int_{-\infty}^{\infty} dp \exp\left(-\gamma\frac{\Lambda}{2}p^2\right) \\ &= h(x_0)\exp(\gamma g(x_0))\sqrt{\frac{2\pi}{\gamma\Lambda}} \end{aligned}$$

This method of deriving asymptotic approximations is known as Laplace's method. It is very widely used. Most of the trouble we encounter in its application is in massaging the integral of interest into the standard form. This is best illustrated by an example, Stirling's approximation to the gamma function

$$\begin{aligned} \Gamma(z) &= \frac{1}{z}\int_0^{\infty} \exp(-t)t^z dt \\ &= \frac{1}{z}\int_0^{\infty} \exp(-t + z\log t)dt \\ &= z^z\int_0^{\infty} \exp(z(\log x - x))dx \end{aligned}$$

So we make the identification $g(s) = \log s - s$ and locate its maximum

$$\begin{aligned} &\frac{dg(x)}{dx} = \frac{1}{x} - 1 \Rightarrow x_0 = 1; \quad g(x_0) = 1 \\ &\frac{d^2g(x)}{dx^2} = -\frac{1}{x^2} \Rightarrow \Lambda = 1 \end{aligned}$$

Plugging all these bits into our standard formula gives us $\Gamma(z) \sim z^{z-1/2}\exp(-z)\sqrt{2\pi}$.

<u>Oscillatory integrals: the method of stationary phase.</u>

In this case we consider integrals of the form

$$F(\gamma) = \int_a^b \exp(i\gamma g(x))h(x)dx; \quad \gamma \to \infty$$

The principal contributions to the integral come from points within the range of integration where the phase function *g* is stationary (i.e. either a maximum or a minimum.) This is to be contrasted with Laplace's method, in which only maxima in *g* contribute. Otherwise it goes through virtually without change

$$\left.\frac{dg(x)}{dx}\right|_{x=x_0} = 0 \text{ (stationary point)}$$

$$g(x_0 + p) \approx g(x_0) - \frac{\Lambda}{2}p^2; \quad \Lambda = -\left.\frac{d^2 g(x)}{dx^2}\right|_{x=x_0}$$

$$\begin{aligned} \mathrm{F}(\gamma) &\sim h(x_0)\exp(i\gamma g(x_0))\int_{-\infty}^{\infty} dp \exp\left(-i\gamma\frac{\Lambda}{2}p^2\right) \\ &= h(x_0)\exp(i\gamma g(x_0))\sqrt{\frac{2\pi}{\gamma|\Lambda|}}\exp\left(-i\frac{\Lambda\pi}{4|\Lambda|}\right) \end{aligned}$$

Differential equations

In the previous session we mentioned that the evolution of many systems of interest can be described in terms of differential equations. While much of classical (Morse and Feshbach) and modern (dynamical systems) physics and engineering focuses on the detailed study of differential equations, we need to know relatively little about them if our principal interest is in the calculation of radar performance. There are, however, a couple of points that we should cover, as they will crop up in our discussions of stochastic processes and tracking. The first thing we look at is the equation for simple relaxation

$$\frac{dy}{dt} = -ay$$

This can be re-arranged and integrates directly

$$\begin{aligned} \frac{dy}{y} &= -adt \\ -at + C &= \log(y) \\ y &= K\exp(-at) \end{aligned}$$

We notice that the solution to this first order differential equation contains a single arbitrary constant, that can be fiddled about with to ensure that *y* satisfies given boundary conditions.
If the relaxing process is further driven by some external influence then the associated differential equation might be

$$\frac{dy}{dt} = -ay + f(t)$$

How might we solve this? If we multiply the whole thing through by $\exp(at)$ and re-arrange to give

$$\frac{d(y\exp(at))}{dt} = \exp(at)f(t)$$

This can be integrated to give us

$$y(t) = y(0)\exp(-at) + \int_0^t \exp(-a(t-t'))f(t')dt'$$

which splits up nicely into a decaying 'transient' and a driven 'response' term. (Solutions of this sort are of particular interest when the relaxing process is perturbed by noise and we consider a Langevin or stochastic differential equation). This simple first order linear ordinary differential equation with a constant relaxation rate (coefficient) can be generalised in many ways. Two cases of particular interest are those in which the order is increased (though the ODE remains linear and with constant coefficients) and in which the relaxation rate varies (the ODE remaining first order and linear). An example of the first case is the equation of motion of a damped harmonic oscillator

$$\frac{d^2x}{dt^2} + K\frac{dx}{dt} + \omega^2 x = 0\ .$$

This can be solved in several ways; substituting a trial solution of the form $x = \exp(\alpha t)$ leads to the quadratic

$$\alpha^2 + K\alpha + \omega^2 = 0$$

with solutions

$$\alpha_\pm = \frac{-K \pm \sqrt{K^2 - 4\omega^2}}{2}\ .$$

This allows us to identify two independent solutions of the 2nd order linear equation; each introduces a constant of integration that can be fixed up by reference to the two boundary conditions required to specify the solution. The inhomogeneous or driven equation

$$\frac{d^2x}{dt^2} + K\frac{dx}{dt} + \omega^2 x = f(t)$$

can be solved in a variety of ways. One of the exercises from the previous session encouraged you to do this using the Laplace transform; an exercise for this session guides you through the application of the 'integrating factor' method we applied earlier to separate out the transient and driven solutions to the first order relaxation process. Another way of looking at this second order differential equation in one dependent variable is as a first order differential in two dependent variables:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -Ky - \omega^2 x$$

In the next session (where we will look at matrices and vectors in some detail) we will have more to say about first order linear vector differential equations of this type.

The other generalisation is to a varying relaxation rate; the differential equation becomes

$$\frac{dy(t)}{dt} + \alpha(t)y(t) = f(t)$$

This is solved by the introduction of an integrating factor

$$\frac{d}{dt}\left(\exp\left(\int^{t} \alpha(t')dt'\right)y(t)\right) = \left(\exp\left(\int^{t} \alpha(t')dt'\right)y(t)\right)f(t)$$

This can now be integrated directly to yield

$$y(t) = y(0)\exp\left(-\int_0^t \alpha(t')dt'\right) + \int_0^t \exp\left(-\int_{t'}^t \alpha(t')dt'\right)f(t')dt'.$$

Unfortunately it is not possible to generalise this integrating factor approach to higher order ODEs with varying coefficients. As we shall see in session 5 (All you wanted to know about flashy sums…) *Mathematica* is a great help in the solution of more challenging differential equations; fortunately we hardly ever come up against them in radar performance and simulation work.

Exercises

We have covered a lot of material this time; consequently there are lots of exercises to help you practise. Don't feel obliged to do them all; pick on the ones related to the material that interested you most. If you think you're hard enough; pick on the ones you suspect that no-one else can do; after all scientists and engineers are un-reconstructedly competitive at heart.

1 Evaluate the first and second derivatives (i.e. dy/dx and d^2y/dx^2) of the following functions

$$y = \sin\left(x^2\right)$$
$$y = \tan\left(\sin^{-1}(x)\right)$$
$$y = \cos(x) + i\sin(x)$$
$$y = \log\left(\sin(x)\right)$$
$$y = 1/\left(1+\cos(x)\right)^2$$

Differentiate the following functions; why are the answers you get so simple? (hopefully)

$$y = \sin^{-1}(x) + \cos^{-1}(x)$$
$$y = \tan^{-1}(x) + \cot^{-1}(x)$$
$$y = \tan^{-1}\left(\frac{a+x}{1-ax}\right)$$

If u and v are functions of x show that

$$\frac{d}{dx}\left(\tan^{-1}(u/v)\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{u^2+v^2}$$

If $y = (\tan x + \sec x)^m$ show that

$$\frac{dy}{dx} = my\sec x$$

What is the nth derivative of

$$y = \frac{x+1}{x^2-4}$$

Remember: divide and conquer (i.e. split it into partial fractions)

In our derivation of derivatives of inverse functions we frequently used

$$dx/dy = (dy/dx)^{-1};$$

show that

$$\frac{d^2x}{dy^2} = -\frac{d^2y/dx^2}{(dy/dx)^3}; \quad \frac{d^3x}{dy^3} = -\left[\frac{d^3y}{dx^3}\frac{dy}{dx} - 3\left(\frac{d^2y}{dx^2}\right)^2\right] \Big/ \left(\frac{dy}{dx}\right)^5$$

Prove that

$$\frac{d}{dx}\left[\cos^{-1}\left(\sqrt{\frac{\cos 3x}{\cos^3 x}}\right)\right] = \sqrt{\frac{3}{\cos x \cos 3x}}\ .$$

(This last one is hard)

2 Evaluate the following indefinite integrals; some of them are straightforward while others require a deal of low cunning:

$$\int \tan(x)dx$$
$$\int \tan^{-1}(x)dx$$
$$\int \tan(x)^2 dx$$
$$\int x^\alpha \log(x)dx$$
$$\int \log(x)^2 dx$$
$$\int \frac{dx}{1-x^2}$$
$$\int \frac{dx}{1+x+x^2+x^3}$$
$$\int \sqrt{1-x^2}\,dx$$
$$\int \sqrt{1+x^2}\,dx$$

Prove that (this one is a bit mean, I'm afraid)

$$\int \frac{dx}{1+x^4} = \frac{1}{4\sqrt{2}}\left\{\log\left(\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2}\right) + 2\tan^{-1}\left(\frac{x\sqrt{2}}{1-x^2}\right)\right\}$$

Evaluate the following definite integrals

$$\int_0^\infty \frac{dx}{1+x^2} \qquad \int_0^\infty \frac{dx}{\left(1+x^2\right)^n}$$

$$\int_0^\infty x^{2n+1}\exp\left(-ax^2\right)dx \qquad \int_0^\infty x^{2n}\exp\left(-ax^2\right)dx$$

$$\int_0^\infty \frac{dx}{\cosh(x)} \qquad \int_0^\infty \frac{dx}{\cosh^4(x)}$$

$$\int_0^\infty \frac{\sin^2 x}{x^2}dx \qquad \int_0^\infty \frac{\sin^3 x}{x^2}dx$$

3 The last time this course was presented in Malvern a challenge was thrown down that, like all challenges, enticed the unwary with the promise of a reward: a bottle of champagne. This bounty went unclaimed, perhaps to be awarded at a later date. Things are different now. Perrier water is all that you can expect for proving the following:

$$\int_0^1 \frac{dx}{x}\left\{\frac{\log((1+x)/2)}{1-x} - \frac{\log((1-x)/2)}{1+x}\right\} = \frac{\pi^2}{12}$$

Nonetheless you could still get the thrill of knowing that you have done something that none of the punters could manage last time. A true *cognoscente* might even track down where this amazing piece of flash comes from (it is related very tenuously to exercise 1 from the previous session); such erudition (along with a satisfactory derivation of the integral) could still be rewarded with a non-PC splash of fizz.

4 One of the perhaps less familiar, but more useful, topics we have covered in this session is the asymptotic approximation to integrals containing a large parameter. The following problems, some of which will find direct application in our radar performance work, will give you some practice

Develop a series approximation to the integral $\int_x^\infty \exp(-t)\frac{dt}{t}$ appropriate to large values of x. Do this by (i) integrating by parts and (ii) using Watson's lemma.

Bessel functions will crop up frequently in subsequent sessions. The zeroth order Bessel functions of the first kind and with real and imaginary arguments can be represented by the definite integrals

$$J_0(z) = \frac{1}{2\pi}\int_0^{2\pi}\exp(iz\cos\theta)d\theta \qquad I_0(z) = \frac{1}{2\pi}\int_0^{2\pi}\exp(z\cos\theta)d\theta$$

Show that, for large z,

$$J_0(z) \sim \sqrt{\frac{2}{\pi z}} \cos(z - \pi/4)$$
$$I_0(z) \sim \sqrt{\frac{1}{2\pi z}} \exp(z)$$

(where does the factor of 2, the most obvious difference between these results, come from?)

The Rice distribution describes the amplitude E of the resultant of a coherent signal (amplitude A) and thermal noise (power x) :

$$P(E|A) = \frac{2E}{x} \exp\left(-\left(E^2 + A^2\right)/x\right) I_0(2EA/x)$$

Using the asymptotic expansion you have just derived show how this reduces to a delta function, centred on A, when the noise power gets small.

Obtain an asymptotic series, appropriate to the large a regime, for the integral

$$\int_0^\infty \exp(-at) \frac{\sin(t)}{t} dt$$

Do this by (i) using Watson's lemma and (ii) evaluating the integral exactly and making a suitable expansion of the result. You should get the same answer from each calculation.

The K distribution will figure in a lot of our radar performance calculations: in its compound form the distribution of the clutter signal intensity z is represented as

$$P(z) = \frac{b^\nu}{\Gamma(\nu)} \int_0^\infty dx \exp(-z/x) x^{\nu-2} \exp(-bx)$$

Using the method of steepest descents investigate the behaviour of this function for large z. How does what you get compare with the large argument behaviour of the underlying exponential distribution of the power of a thermal noise process?

5 Perhaps you would like to follow through the solution of the driven second order linear ODE with constant coefficients, extending the integrating factor method. Hints are provided. We consider the inhomogeneous case satisfying

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = h(t).$$

'Factorise' the differential operator to give us

$$\left(\frac{d}{dt} - \alpha_+\right)\left(\frac{d}{dt} - \alpha_-\right) y = h(t)$$

Rewrite this as

$$\left(\frac{d}{dt}-\alpha_+\right)u = h(t)$$
$$\left(\frac{d}{dt}-\alpha_-\right)y = u$$

The first of these inhomogeneous first order ODEs can be solved for *u*, this is then plugged into the second ODE, which is solved to give us the required solution. Check that this procedure leads to

$$y(t) = y_0\exp(\alpha_- t) + \frac{u_0\left(\exp(\alpha_+ t)-\exp(\alpha_- t)\right)}{(\alpha_+-\alpha_-)} + \exp(\alpha_- t)\int_0^t \exp((\alpha_+-\alpha_-)t')\left(\int_0^{t'}\exp(-\alpha_+ t'')h(t'')dt''\right)dt'$$

and that a final integration by parts reduces this to

$$y(t) = y_0\exp(\alpha_- t) + \frac{u_0\left(\exp(\alpha_+ t)-\exp(\alpha_- t)\right)}{(\alpha_+-\alpha_-)} + \frac{1}{(\alpha_+-\alpha_-)}\int_0^t\left(\exp(\alpha_+(t-t'))-\exp(\alpha_-(t-t'))\right)h(t')dt'$$

Compare this with what you might have done in response to exercise 3 in session 1.

6 The following is a 'real' problem, whose solution uses the integrating factor technique we discussed in this session. A quantity $\phi(\mathbf{k},\mathbf{x},t)$ satisfies the relaxation equation

$$\frac{d\phi(\mathbf{k},\mathbf{x},t)}{dt} = \beta(t)\left[\phi(\mathbf{k},\mathbf{x},t) - \frac{\phi(\mathbf{k},\mathbf{x},t)^2}{\phi_0(\mathbf{k})}\right]$$

where $\mathbf{x},\mathbf{k}$ evolve subject to the equations

$$\frac{d\mathbf{x}}{dt} = \nabla_{\mathbf{k}}\omega_0(\mathbf{k}) + \mathbf{U},$$
$$\frac{d\mathbf{k}}{dt} = -\mathbf{k}.\nabla_{\mathbf{x}}\mathbf{U}.$$

It is assumed that you know β and ϕ_0. Employ an integrating factor to solve the relaxation equation, then fiddle the resulting solution about until you have

$$\varphi(t) = \frac{\varphi_0(t)}{1-\varphi_0(t)\int_{t_0}^t dt'\frac{1}{\varphi_0(t')^2}\nabla_{\mathbf{k}}\varphi_0(\mathbf{k}).\left[\mathbf{k}.\nabla_{\mathbf{x}}\mathbf{U}\right]_{x=x(t')}\exp\left(-\int_{t'}^t dt''\beta(t'')\right)}$$

This problem is simple in principle, but a bit messy in detail. (It is part of an analysis of the modulation of sea surface roughness by a current)