Fourier Series and Integral Transforms

Introduction

In today's sessions we will be concerned with the representation and analysis of functions in terms of series of, and integral representations derived from, relatively simple basis functions. The representation of a function over a finite range of its argument as a sum of sines and cosines (i.e. a Fourier series) provides us with a familiar starting point. The Fourier transform (FT), in which the sum over discrete frequencies in the Fourier series is replaced by an integral over a continuum, emerges as the range of the expanded function's argument becomes infinite. The trigonometric building blocks of the Fourier series and transform have very simple properties (notably $\exp(ikx)\exp(ikx') = \exp(ik(x + x'))$) reflected in the properties of the Fourier transform itself, which make it so useful in many practical calculations. Viewed in the context of the theory of functions of a complex variable the Laplace transform (LT) and its inversion can be seen to equivalent to the Fourier transform pair. The relationship between the FT and LT and the rather less familiar Mellin transform also becomes clear. The sines and cosines that play a central role in Fourier analysis emerge as solutions of the one-dimensional wave equation; their two- and three-dimensional analogues are the Bessel functions. Our review of integral transforms is thus concluded by a quick review of Hankel and related transforms. Any paper and pencil evaluation or inversion of these transforms is an exercise in integral calculus, for which the previous sessions should stand us in good stead. Failing that, one can make reference to any one of several comprehensive tables, which record the labours of transformers and inverters of bye-gone days. Should we, and they, be unable to evaluate the transform analytically, a numerical approach is needed; this is where the Fast Fourier Transform algorithm comes into its own. The remarks apply with yet greater force when we come to apply Fourier methods in data analysis. As the FFT underpins much of radar and related signal processing, we discuss it, and related issues of sampling and interpolation, in some detail. An outline of the intimate connection between Fourier analysis and synthesis and radar image formation brings things to a close.

Trigonometric series; an example of expansion in orthogonal functions

If we wish to represent a function $f(x)$ defined on the interval $-\pi \le x \le \pi$ we first consider the functions $\sin(nx), \quad \cos(nx)$, where *n* is an integer. These obey the integral relations (that are readily verified)

$$\begin{gathered}\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx = 0 \\ \int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = \int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = \pi\delta_{m,n}\end{gathered} \qquad (1)$$

We now represent the function as a sum of 2*N* +1 sine and cosine terms:

$$\Omega_N(x) = \sum_{n=0}^{N}\left(a_n \sin(nx) + b_n \cos(nx)\right) \approx f(x) \quad -\pi \le x \le \pi \qquad (2)$$

We identify the coefficients *a*,*b*, as those that minimise the 'least square' error

$$\int_{-\pi}^{\pi}\left(\Omega_N(x)-f(x)\right)^2 dx = \left(2\pi b_0^2 - 2b_0\int_{-\pi}^{\pi} f(x)dx\right)_0 + \sum_{n=1}^{N}\left(\pi\left(a_n^2+b_n^2\right) - 2a_n\int_{-\pi}^{\pi} f(x)\sin(nx)dx - 2b_n\int_{-\pi}^{\pi} f(x)\cos(nx)dx + \int_{-\pi}^{\pi} f(x)^2 dx\right) \qquad (3)$$

The condition that this mean square error be minimised is found by differentiating with respect to the *a*s and *b*s. This gives us

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \quad n \geq 1$$
$$b_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \qquad (4)$$

We note that the expressions for the 'best' coefficients do not depend on *N*. If we let *N* go 'to infinity' we identify the trigonometric expansion with the Fourier series representation of *f*. It is also possible to express the Fourier series in terms of complex exponentials as follows

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp(inx); \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(-inx)f(x)dx \qquad (5)$$

The evaluation of the Fourier coefficients *a*, *b* and *c* is simply a matter of integration; if *f* is an elementary function this should present no significant problems. The question arises as to what sort of functions can be represented as Fourier series. This problem occupied pure mathematicians for centuries. The fruits of their labours need not concern us here, as they were mainly concerned with functions that were relatively badly behaved. If *f* is continuous and the integrals defining the Fourier coefficients exist there is no problem. One slightly pathological case that is of interest is where the function shows a discontinuity. As we shall see the Fourier series has some difficulty in representing functions of this type. We consider the example

$$\begin{aligned} &f(x) = -1, \quad -\pi \leq x < 0 \\ &f(0) = 0 \\ &f(x) = 1 \quad 0 < x \leq \pi \end{aligned} \qquad (6)$$

This is a step function. As it is an odd function we can expand it in terms of sines alone; the Fourier coefficients are given by

$$a_n = \frac{2}{\pi}\int_0^{\pi}\sin(nx)dx = \frac{2}{\pi n}\left(1-\cos(n\pi)\right) = \frac{2}{n\pi}\left(1-(-1)^n\right) \qquad (7).$$

The Fourier series representation of the discontinuous step function is obtained from

$$\Omega_N(x) = \frac{4}{\pi}\sum_{n=0}^{N}\frac{\sin((2n+1)x)}{2n+1} \qquad (8)$$

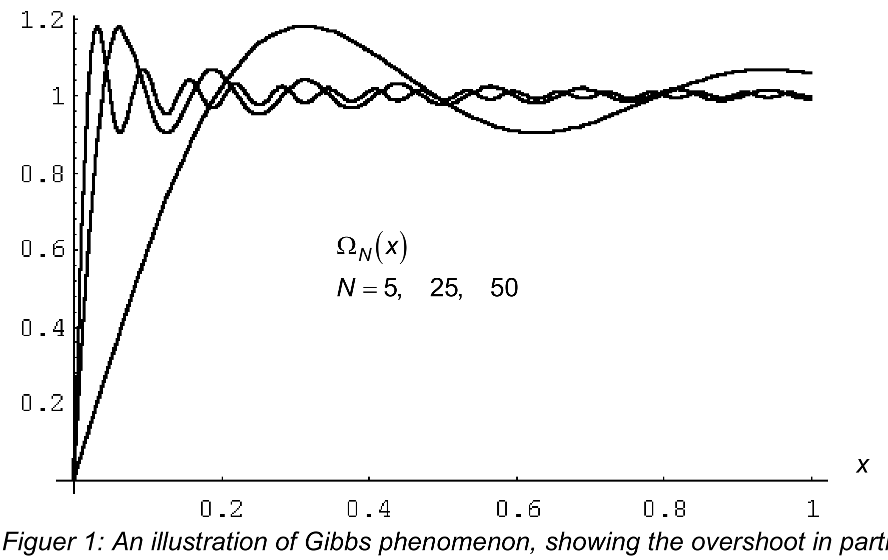
by letting *N* increase without limit. How does this behave for big *N*? To get a handle on this problem we have to use a little low cunning (the same trick will come in handy when you have a go at the first of the exercises). We represent each term in the Fourier series as an integral:

$$\frac{\sin((2n+1)x)}{2n+1} = \int_0^x \cos((2n+1)x')dx' \tag{9}$$

so that

$$\begin{aligned}\Omega_N(x) &= \frac{4}{\pi}\int_0^x \sum_{n=0}^{N} \cos((2n+1)x')dx' \\ &= \frac{2}{\pi}\int_0^x \frac{\sin(2(N+1)x')}{\sin(x')}dx' \\ &\approx \frac{2}{\pi}\int_0^{2(N+1)x} \frac{\sin(p)}{p}dp\end{aligned} \tag{10}$$

We can plot this function out and observe an overshoot of about 18% in $\Omega_N(x)$, that gets closer to the origin, and ever more narrow, as *N* gets larger, but is always present and always of the same magnitude. This overshooting in the representation of a discontinuity by a trigonometric series of a finite number of terms is a general occurrence. It is referred to as Gibb's phenomenon, after the chap who first noticed and explained it.



Figuer 1: An illustration of Gibbs phenomenon, showing the overshoot in partial sums of the Fourier series.

Expansion in terms of other orthogonal sets of functions

The procedure we have just adopted, of expanding an arbitrary function in terms of sines and cosines with coefficients given in terms of integrals of a product of this function and the corresponding sine or cosine, can be extended to other sets of 'basis' functions. These in general satisfy the orthogonality relation

$$\int_{x_1}^{x_2} \phi_m(x)^* \phi_n(x) w(x) dx = \delta_{n,m} \tag{11}$$

where x_1, x_2 are the limits of the range over which the expanded function is defined and $w(x)$ is a weighting function. The function *f* is then expanded as a sum of these orthogonal functions

$$f_N(x) = \sum_{n=0}^{N} a_n \phi_n(x) \tag{12}$$

If we now define a weighted mean square error through

$$\Delta_N = \int_{x_1}^{x_2} w(x) \left| f(x) - f_N(x) \right|^2 \tag{13}$$

then the conditions that this be minimised with respect to the coefficients

$$\frac{\partial \Delta_N}{\partial a_n^*} = 0; \quad \frac{\partial \Delta_N}{\partial a_n} = 0 \tag{14}$$

are satisfied by

$$a_n = \int_{x_1}^{x_2} w(x) f(x) \phi_n(x)^* dx\ . \tag{15}$$

This mean square error condition is again seen to be independent of *N*. Some examples of orthogonal functions, with their corresponding weighting functions and supports, are as follows:

Legendre polynomials:

$$x_1 = -1, \quad x_2 = 1, \quad w(x) = 1; \quad \phi_n(x) = \sqrt{\frac{2}{2n+1}} P_n(x); \quad P_n(x) = {}_2F_1\left(-n, n+1; 1; (1-x)/2\right) \tag{16}$$

Generalised Laguerre polynomials

$$\begin{gathered} x_1 = 0, \quad x_2 = \infty, \quad w(x) = x^\alpha \exp(-x), \quad \phi_n(x) = \sqrt{\frac{n!}{\Gamma(n+\alpha+1)}}\, L_n^{(\alpha)}(x) \\ L_n^{(\alpha)}(x) = \frac{\Gamma(\alpha+n+1)}{n!\,\Gamma(\alpha+1)}\, {}_1F_1(-n; \alpha+1; x) \end{gathered} \tag{17}$$

Hermite polynomials

$$\begin{gathered} x_1 = -\infty, \quad x_2 = \infty, \quad w(x) = \exp\left(-x^2\right), \quad \phi_n(x) = \frac{1}{\sqrt{2^n n! \pi^{1/2}}} H_n(x) \\ H_{2n}(x) = (-1)^n 2^{2n} n!\, L_n^{(-1/2)}\left(x^2\right); \quad H_{2n+1}(x) = (-1)^n 2^{2n+1} n!\, x L_n^{(1/2)}\left(x^2\right) \end{gathered} \tag{18}$$

As we shall see in the next session, Mathematica allows us to access and manipulate the fancy stuff that crops up here. You might well have made the acquaintance of these classical orthogonal polynomials when studying quantum mechanics, where the concept of an expansion in a series of eigenfunctions of the Hamiltonian is widely used. The Hermite and Laguerre polynomials also crop up in the analysis of correlated Gaussian- and gamma –Markof-distributed random variables; rather remarkably their structures effectively encode the correlation properties of these quantities. Series expansions of this type display the qualitative properties of the more familiar Fourier series, including the occurrence of Gibbs phenomenon.

The Fourier transform

So far we have considered the Fourier expansion of a function over the range $-\pi \le x \le \pi$, an analogous expansion can be made over the range $-L/2 \le x \le L/2$ as

$$f(x) = \frac{1}{L}\sum_{n=-\infty}^{\infty} \exp(in\,2\pi x/L) \int_{-L/2}^{L/2} f(x')\exp(-in2\pi x'/L)dx' \tag{19}$$

where we have chosen the exponential form of the FT as it is more compactly represented. As L gets larger the separation between the frequencies of the constituent trig functions gets smaller and smaller; thus in the limit of very large (and ultimately infinite) L we would expect a continuum of frequencies to contribute. The sum in the trigonometric series would then become an integral and we would have a Fourier transform:

$$\begin{gathered} k_n = \frac{n2\pi}{L}, \Delta k_n = \frac{2\pi}{L} \\ f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \exp(ik_n x)\tilde{f}(k_n, L)\Delta k_n; \quad \tilde{f}(k_n, L) = \int_{-L/2}^{L/2} \exp(-ik_n x')f(x')dx' \end{gathered} \tag{20}$$

As $L \to \infty$ the summation can be rendered as an integration, leading to

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \exp(ikx)\tilde{f}(k); \quad \tilde{f}(k) = \int_{-\infty}^{\infty} dx \exp(-ikx)f(x) \tag{21}$$

It is common practice to think in terms of Dirac delta 'functions' that have the properties

$$\begin{gathered} \delta(x) = 0; \quad x \ne 0 \\ \int_{-\varepsilon}^{\varepsilon} \delta(x)dx = 1 \\ \int_{-\infty}^{\infty} f(x')\delta(x'-x)dx' = f(x) \end{gathered} \tag{22}$$

The Fourier transform theorem can then be expressed as follows:

$$\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \exp(ik(x-x')). \tag{23}$$

Having identified the delta function in this form we see that the Fourier transformation of a periodic function, which we can represent as a Fourier series, is an infinite sum of delta functions in frequency space:

$$\begin{aligned} f(x) &= f(x+L) \\ &= \sum_{n=-\infty}^{\infty} a_n \exp\left(\frac{2\pi i n x}{L}\right); \quad a_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\exp\left(-\frac{2\pi i n x}{L}\right)dx \end{aligned} \tag{24}$$

$$\tilde{f}(k) = 2\pi \sum_{n=-\infty}^{\infty} a_{-n}\delta\left(k - \frac{2n\pi}{L}\right)$$

This result can be captured formally through the FT of a sum of equally spaced delta functions

$$\delta_p(x,L) = \sum_{n=-\infty}^{\infty} \delta(x-nL); \quad \tilde{\delta}_p(k,L) = \frac{2\pi}{L}\sum_{n=-\infty}^{\infty}\delta\left(k - \frac{2n\pi}{L}\right) \tag{25}$$

It is also possible to make the Fourier transform theorem quite credible without making reference to the Fourier series and going to the infinite *L* limit. First we note some examples of Fourier transforms (which you may like to prove)

$$\begin{aligned} &f(x) = 1, |x| \le a; = 0 \text{ otherwise} && \tilde{f}(k) = \frac{2\sin(ka)}{k} \\ &f(x) = \exp(-a|x|) && \tilde{f}(k) = \frac{2a}{a^2+k^2} \\ &f(x) = \exp(-ax^2) && \tilde{f}(k) = \exp\left(-k^2/4a\right)\sqrt{\frac{\pi}{a}} \end{aligned} \tag{26}$$

Any of the results (26) can be used to make the fundamental FT theorem plausible. Let us consider

$$\begin{aligned} \Delta_a(x-x') &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \exp(ik(x-x'))\exp(-ak^2) \\ &= \frac{1}{2\sqrt{\pi a}}\exp\left(-(x-x')^2/4a\right) \end{aligned} \tag{27}$$

Considered as a function of *x*, this is peaked around *x*' and has a unit area. As *a* gets small the peak value of the function gets progressively larger, while the function gets narrower and narrower, but is still centred on *x*'. Thus $\Delta_a(x-x')$ behaves like the Dirac delta function, so that

$$\lim_{a\to 0}\int_{-\infty}^{\infty} dx f(x)\Delta_a(x-x') = f(x') \tag{28}$$

As a goes to zero the Gaussian weighting in the definition of $\Delta_a(x-x')$ tends to unity, and the weighting becomes uniform, so that

$$\begin{aligned}\delta(x-x') &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \exp(ik(x-x')) \\ f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx' f(x')\int_{-\infty}^{\infty} dk \exp(ik(x-x'))\end{aligned} \tag{29}$$

which is equivalent to the Fourier integral theorem. You might like to check that the other FTs we have just evaluated lead to the same conclusion, via different 'models' for the delta function. Rigorous proofs can be based on arguments like these, if required.

Many properties of the FT are derivable from the definitions we have just given, and a bit of calculus; as we mentioned earlier these useful results reflect the simple properties of the exponential function:

$$\begin{aligned}&\exp(ikx)\exp(ikx') = \exp(ik(x+x')) \\ &\frac{d\exp(-ikx)}{dx} = -ik\exp(-ikx)\end{aligned} \tag{30}$$

So we have

The shift theorem:

$$\int_{-\infty}^{\infty} dx \exp(-ikx) f(x+a) = \exp(ika)\tilde{f}(k) \tag{31}$$

If $f(x)$ is real then

$$\tilde{f}(k)^* = \left(\int_{-\infty}^{\infty} f(x)\exp(ikx)dx\right)^* = \tilde{f}(-k) \tag{32}$$

From this it follows that, if f is an even (odd) function of x, then $\tilde{f}$ is a real (imaginary).

The derivative theorem, proved by integrating by parts

$$\int_{-\infty}^{\infty} dx \exp(-ikx)\frac{df(x)}{dx} = ik\tilde{f}(k) \tag{33}$$

Substituting appropriate Fourier representations into the integrand and identification of the delta function (23) lead us to:

The convolution theorem: if

$$g \otimes h(x) = \int_{-\infty}^{\infty} g(x - x')h(x')dx' \text{ then } \int_{-\infty}^{\infty} \exp(-ikx) g \otimes h(x)dx = \tilde{g}(k)\tilde{h}(k) \tag{34}$$

$$\int_{-\infty}^{\infty} \exp(ikx) f(x) g(x) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega - \omega')\tilde{g}(\omega')d\omega' = \frac{1}{2\pi}\tilde{f} \otimes \tilde{g}(\omega) \tag{35}$$

and Parseval's theorem

$$\int_{-\infty}^{\infty} dk \left|\tilde{f}(k)\right|^2 = \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \exp(ik(x - x')) f(x) f(x')^* = 2\pi \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \delta(x - x') f(x) f(x')^* = 2\pi \int_{-\infty}^{\infty} dx \left|f(x)\right|^2 \tag{36}$$

(The power in a signal is conserved during Fourier transformation.)

We should also mention the 'uncertainty principle'. This quantifies the qualitative observation, illustrated by the results in (26), that the FT of a localised function is itself spread out, and *vice versa*. A measure of the extent to which a function is localised in real space is provided by

$$(\Delta x)^2 = \frac{\int (x - \bar{x})^2 |f(x)|^2 dx}{\int |f(x)|^2 dx}, \quad \bar{x} = \frac{\int x |f(x)|^2 dx}{\int |f(x)|^2 dx}; \tag{37}$$

the corresponding measure of localisation in the frequency domain is

$$(\Delta k)^2 = \frac{\int (k - \bar{k})^2 \left|\tilde{f}(k)\right|^2 dk}{\int \left|\tilde{f}(k)\right|^2 dk}, \quad \bar{k} = \frac{\int k \left|\tilde{f}(k)\right|^2 dk}{\int \left|\tilde{f}(k)\right|^2 dk} \tag{38}$$

These satisfy

$$(\Delta x)^2 (\Delta k)^2 \geq \frac{1}{4} \tag{39}$$

which reduces to an equality when f and its transform take the Gaussian form. This result is equivalent to the Heisenberg uncertainty relation, familiar to students of quantum mechanics who will also recall that the ground state wave function of the simple harmonic oscillator is a Gaussian function. To prove (39) we need Schwartz' inequality, which can be derived as follows: We see that, for two, possibly complex, functions of x, f and g, and a real parameter λ, we have

$$\int |f + \lambda g|^2 dx = \int |f|^2 dx + \lambda^2 \int |g|^2 dx + \lambda \int (f^* g + g^* f) dx \geq 0 \tag{40}$$

As this quantity is always greater that zero, the quadratic equation in λ

$$\int |f|^2 dx + \lambda^2 \int |g|^2 dx + \lambda \int (f^* g + g^* f) dx = 0 \tag{41}$$

cannot have real roots i.e.

$$\left(\int (f^* g + g^* f)dx\right)^2 \le 4\int |f|^2 dx \int |g|^2 dx \tag{42}$$

This is Schwartz inequality; we now apply it to the two functions

$$(x-\bar{x})f(x), \quad df(x)/dx + i\bar{k}f(x) \tag{43}$$

We find that

$$\left(\int (x-\bar{x})\frac{d|f(x)|^2}{dx}dx\right)^2 \le 4\int (x-\bar{x})^2 |f(x)|^2 dx \int \left(f'(x) + i\bar{k}f(x)\right)\left(f'(x)^* - i\bar{k}f(x)^*\right)dx \tag{44}$$

Integrating by parts gives us

$$\int (x-\bar{x})\frac{d|f(x)|^2}{dx}dx = -\int |f(x)|^2 dx\,, \tag{45}$$

while the application of Parseval's theorem to $\int \left(f'(x) + i\bar{k}f(x)\right)\left(f'(x)^* - i\bar{k}f(x)^*\right)dx$ relates it to $(\Delta k)^2$. When we tidy things up we find that (39) pops out.

We also see from the FTs given in (26) that, if *f* is smooth, $\tilde{f}(k)$ is killed of rapidly, while discontinuities in the function or its derivatives induce a much slower, power law, decay at high frequencies. Results of this type are known as Tauber theorems; rather than pursue these in detail here, we will be content with the simple observation that the behaviour of the $\tilde{f}(k)$ at large *k* senses the small scale structure of *f*, and *vice versa*.

The concept of the FT is readily extended to two, three and more dimensions; the properties we have just outlined still hold, suitably generalised.

$$\begin{aligned}\tilde{f}(\mathbf{k}) &= \int d^3x \exp(-i\mathbf{k}.\mathbf{x}) f(\mathbf{x}) \\ f(\mathbf{x}) &= \frac{1}{(2\pi)^3}\int d^3x \exp(i\mathbf{k}.\mathbf{x})\tilde{f}(\mathbf{k})\end{aligned} \tag{46}$$

$$\delta(\mathbf{x}-\mathbf{x}') = \frac{1}{(2\pi)^3}\int d^3k \exp(-i\mathbf{k}.(\mathbf{x}-\mathbf{x}')) \tag{47}$$

$$\int d^3k \exp(i\mathbf{k}.\mathbf{x})\exp(i\mathbf{k}.\mathbf{x}_0)\tilde{f}(\mathbf{k}) = \int d^3k \exp\left(i\mathbf{k}.(\mathbf{x}+\mathbf{x}_0)\right)\tilde{f}(\mathbf{k}) = (2\pi)^3 f(\mathbf{x}+\mathbf{x}_0) \tag{48}$$

$$\int d^3x \exp(-i\mathbf{k}.\mathbf{x})\nabla f(\mathbf{x}) = i\,\mathbf{k}\int d^3x \exp(-i\mathbf{k}.\mathbf{x}) f(\mathbf{x}) \tag{49}$$

We also have the so-called Fourier slice theorem:

Inversion of a 3-dimensional Fourier transform $\tilde{f}(\mathbf{k})$ with respect to two components (e.g. k_x, k_y) of the wave vector **k** gives an image of $f(\mathbf{x})$, projected onto the plane containing those two wave vector components
(e.g. the *x*-*y* plane)

$$\begin{aligned}&\frac{1}{(2\pi)^2}\int dk_x\int dk_y\tilde{f}(k_x,k_y)\exp\left(i(xk_x+yk_y)\right)=\\&\qquad\frac{1}{(2\pi)^2}\int dk_x\int dk_y\exp\left(i(xk_x+yk_y)\right)\int dx'dy'dz'\exp\left(-i(x'k_x+y'k_y)\right)f(x',y',z')\end{aligned}\tag{50}$$

If we now identify a pair of Dirac delta functions we find that this reduces to

$$\int dx'dy'dz'f(x',y',z')\delta(x-x')\delta(y-y')=\int dz'f(x,y,z')\tag{51}$$

The Laplace transform

The FT has been developed in the context of functions defined over the range $-\infty < x < \infty$ and for which the defining integral $\tilde{f}(k)=\int_{-\infty}^{\infty}dx\exp(-ikx)f(x)$ exists. Frequently these conditions do not apply; lots of things start at a given instant ($t=0$) and generate transients that are of interest while quite simple functions may have FTs that are, well, non-exixstent e.g. $f(x)=x^2,\exp(ax)$. The Laplace transform is designed to take care of these problems. The first bit is easy; if the function of interest is zero before time *t*, have your integral transform running only over positive *t*. To accommodate functions for which this half range FT does not converge we stick in a convergence factor (that can also be thought of as an imaginary part to the frequency *k*) So, first of all, we introduce notation for the half-range FT:

$$\tilde{f}_>(k)=\int_0^{\infty}dx\exp(-ikx)f(x)\tag{52}$$

In terms of this we have

$$\int_0^{\infty}dx\exp(-ikx)\exp(-\gamma x)f(x)=\tilde{f}_>(k-i\gamma)\tag{53}$$

From the Fourier inversion theorem we now have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\exp(ikx)\tilde{f}_>(k-i\gamma)&=\exp(-\gamma x)f(x)\quad x\ge 0\\&=0\quad x<0\end{aligned}\tag{54}$$

So

$$f(x)\Theta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(i(k-i\gamma)x\right)\tilde{f}_{>}(k-i\gamma)dk$$
$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\exp(sx)\,F(s)ds \tag{55}$$

Here we have made the change of variable $s = i(k - i\gamma)$ and identified the Laplace transform as

$$F(s) = \int_0^{\infty}\exp(-sx)f(x)dx\ . \tag{56}$$

In elementary discussions the LT is frequently introduced via this definition; inversion of the LT is then performed by reference to a table of transforms. Here we have seen that the LT and FT are very closely related. Armed with our Laplace inversion formula and some practice in integrating in the complex plane, we might hope to dispense with the table of transforms. In cases where the inversion integral cannot be evaluated explicitly we can also make use of asymptotic and other techniques to get an approximate but useful answer out. However we will not pursue this in any detail here; should any of you feel the need, a table of transforms will give you plenty to work out on.

The Laplace transform has a set of properties analogous to those of the Fourier transform

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\exp(sx)\exp(sa)F(s)ds = f(x+a)\Theta(x+a)$$
$$\int_0^{\infty}dx\exp(-sx)\int_0^{x}g(x-x')h(x')dx' = G(s)H(s) \tag{57}$$
$$\int_0^{\infty}dx\exp(-sx)\frac{df(x)}{dx} = sF(s) - f(0)$$

We note the step function that is carried round in the LT 'shift' theorem and the rather different definition of the convolution that turns up here (This is a consequence of *g* and *h* both being defined only for positive arguments; for $x' > x \quad h(x-x')$ vanishes.) While it is possible to extend the definition of the LT to more than one variable this will not be required for the application we are interested in.

The Mellin transform

In the previous section we were able to convert the Fourier transform into the Laplace transform by allowing the frequency *k* to take a complex value then making a change in variable in the integration. We can also modify the Fourier transform to produce the Mellin transform. Just as in the discussion of the Laplace transform we allow the frequency *k* to become complex, to assist in the convergence of the Fourier integral; this trick has its counterpart in the shifting of the contour of the Fourier inversion integral away from the real axis:

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \exp(-ikx) f(x) dx; \quad -\alpha < \Im(k) < -\beta$$
$$f(x) = \frac{1}{2\pi} \int_{i\gamma-\infty}^{i\gamma+\infty} \exp(ikx) \tilde{f}(k) dk \quad -\alpha < \gamma < -\beta \tag{58}$$

If in the first of these we make the identifications

$$p = ik$$
$$s = \exp(-x) \tag{59}$$
$$g(s) = f(\log(s))$$

then the FT pair become

$$G_M(p) = \int_0^{\infty} s^{p-1} g(s) ds$$
$$g(s) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} s^{-p} G_M(p) dp \tag{60}$$

which are commonly referred to as the Mellin transform pair. Our principal interest in the Mellin transform is in its convolution property, which as we shall see in the exercises, is ideally suited to the analysis of the compound form of the K distribution. The 'Mellin' convolution of two functions is defined by

$$g *_M h(s) = \int_0^{\infty} g(s/u) h(u) \frac{du}{u} \tag{61}$$

If we now take the Mellin transform of this we obtain

$$\int_0^{\infty} s^{p-1} \left(\int_0^{\infty} g(s/u) h(u) \frac{du}{u} \right) ds = \int_0^{\infty} du \int_0^{\infty} dt (ut)^{p-1} g(t) h(u)$$
$$= G_M(p) H_M(p) \tag{62}$$

where we have noted that under the change in variables

$$(s,u) \to (t,u); t = s/u$$
$$dsdu \to ududt \tag{63}$$

as can be verified explicitly by calculating the Jacobian. Thus the Mellin transform of a Mellin convolution is the product of the Mellin transforms of the individual functions that make up the convolution. We find this useful in analysing the compound model of non-Gaussian sea clutter.

The Hankel transform

The final integral transform we look at is the Hankel transform; this is closely related to the two-dimensional Fourier transform. Consider a function in 2D that depends only on $r=\sqrt{x^2+y^2}$, and construct its Fourier transform

$$\tilde{f}\left(k_x,k_y\right)=\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\exp\left(i\left(k_x x+k_y y\right)\right)f\left(\sqrt{x^2+y^2}\right) \tag{64}$$

Adopting polar coordinates we write this as

$$\tilde{f}(k)=\int_0^{\infty}drrf(r)\int_0^{2\pi}\exp(ikr\cos\theta)d\theta=2\pi\int_0^{\infty}drrf(r)J_0(kr) \tag{65}$$

Here we have identified the Bessel function through its integral representation (the usual way in which it crops up in radar applications; the I,Q components of the signal span a two dimensional plane)

$$J_0(z)=\frac{1}{2\pi}\int_0^{2\pi}\exp(iz\cos\theta)d\theta \tag{66}$$

Because the **x** space function is isotropic, so is its FT, which depends only on the amplitude *k* of the **k** vector. The original function can be recovered by Fourier inversion; when this is performed in polar co-ordinates another Bessel function emerges to give us

$$f(r)=\frac{1}{(2\pi)}\int_0^{\infty}dkkJ_0(kr)\tilde{f}(k) \tag{67}$$

Implicit in (65) and (67) are the Hankel transform pair

$$\tilde{f}_H(k)=\int_0^{\infty}drrf(r)J_0(kr);\quad f(r)=\int_0^{\infty}dkkJ_0(kr)\tilde{f}_H(k) \tag{68}$$

and the representation of the delta function

$$\frac{\delta(r-r')}{r}=\int_0^{\infty}dkkJ_0(kr)J_0(kr'). \tag{69}$$

Out old friend, the FT of a Gaussian appears in a slightly different guise

$$\int_0^{\infty}drrJ_0(kr)\exp\left(-\alpha r^2\right)=\frac{1}{2\alpha}\exp\left(-k^2/4\alpha\right). \tag{70}$$

Sampling and interpolation

So far we have discussed functions whose value is known for all values of its argument. If we wish to apply these methods to measured data we immediately encounter a problem: we will only know the value of the function of interest at the points where we sample it. To what extent can we apply the ideas we have developed to such a collection of data? (From now on we will work with time series, as they are easy to get an intuitive grip on; so $x \to t; k \to \omega, L \to T$ in our earlier discussion of Fourier series and transformations.)

We consider a function $f(t)$, which we sample at regular intervals of Δ. The act of sampling effectively replaces *f* by

$$f_s(t) = f(t)\sum_{n=-\infty}^{\infty}\delta(t - n\Delta) = f(t)\delta_p(t,\Delta) \tag{71}$$

(We envisage the sampling process at $t = n\Delta$ to be analogous to integration over a short interval around that time.) What is the Fourier transform of this sampled function? Using the convolution theorem and the FT of an infinite train of impulses (25) we find that

$$\tilde{f}_s(\omega) = \frac{1}{\Delta}\sum_{n=-\infty}^{\infty}\tilde{f}(\omega - 2\pi n/\Delta). \tag{72}$$

Thus far we have just been twiddling formulae; is this result of any practical use? When the function *f* is smooth we can expect its FT to be killed off for $|\omega| > \omega_B$; such a function is said to be band limited. So, as long as $\omega_B < \pi/\Delta$, there is no significant overlap between adjacent components in the sum (72). Thus we can recover the FT of band limited function from that of the sampled function

$$\tilde{f}(\omega) = \Delta\tilde{f}_s(\omega), \quad |\omega| < \omega_B \tag{73}$$

as long as the sampling interval Δ satisfies

$$\Delta \le \frac{\pi}{\omega_B}. \tag{74}$$

Sampling at the rate at which the equality holds is said to be at the Nyquist frequency; more frequent sampling is said to be super-Nyquist. The function *f* can now be reconstructed by Fourier inversion:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(\omega)\exp(-i\omega t)d\omega = \frac{1}{2\pi}\int_{-\omega_B}^{\omega_B}\tilde{f}(\omega)\exp(-i\omega t)d\omega = \frac{\Delta}{2\pi}\int_{-\omega_B}^{\omega_B}\tilde{f}_s(\omega)\exp(-i\omega t)d\omega\,. \tag{75}$$

If we now introduce

$$\tilde{f}(\omega)_s = \sum_{n=-\infty}^{\infty} f(n\Delta)\exp(i\omega n\Delta) \tag{76}$$

we find that

$$\begin{aligned} f(t) &= \frac{\Delta}{2\pi}\sum_{n=-\infty}^{\infty} f(n\Delta)\int_{-\omega_B}^{\omega_B} \exp(i(n\Delta - t)\omega)d\omega \\ &= \frac{\Delta}{\pi}\sum_{n=-\infty}^{\infty} f(n\Delta)\frac{\sin((t-n\Delta)\omega_B)}{(t-n\Delta)} \end{aligned} \tag{77}$$

If *f* is localised then this reduces to a sum with a finite number of terms; should we sample at the Nyquist frequency we have

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{\omega_B}\right)\frac{\sin(\omega_B t - n\pi)}{(\omega_B t - n\pi)} \\ &= \sum_{n=-\infty}^{\infty} f(n\Delta_N)\frac{\sin(\pi(t/\Delta_N - n))}{\pi(t/\Delta_N - n)}; \quad \Delta_N = \frac{\pi}{\omega_B} \end{aligned} . \tag{78}$$

This process is referred to as band limited or sinc interpolation.

Should we sample at a rate less than Nyquist then the $n = 0$ term in (72) is contaminated by higher order terms and a reconstruction of the form

$$f(t) = \sum_{n=-\infty}^{\infty} f(n\Delta)\frac{\sin(\pi(t/\Delta - n))}{\pi(t/\Delta - n)} \tag{79}$$

will be corrupted. This phenomenon is known as aliasing.

If, as might well happen in practice, *f* continues to rattle about over the period that it is measured one is in effect pre-multiplying it by a window or weighting function. Consequently the FT obtained through (76) will be convolved with that of the window, and so may be spread out significantly.. Thus the choice of weighting function can be important: the discontinuous all or nothing, top-hat, weighting has a very slowly decaying FT (see (26)) that can itself give rise to aliasing if care is not taken. In practice, this problem is ameliorated by the use of an appropriately tailored weighting, or by super Nyquist sampling.

The Fast Fourier transform (FFT)

So far we have looked at the Fourier transform from a formal viewpoint, working things out by hand, or not at all. However, the decomposition a signal into its harmonic components and identification of their amplitudes plays a central role in much of radar and other signal processing. Doppler processing is an obvious example of this. Were it not for the availability of an efficient method for doing this, these applications might well not be feasible in practice.

First, we consider the discrete Fourier transform (DFT) of a set of *N* data, $a_n, n = 0, \cdots, N$, which might, perhaps be measurements of a time series, sampled at equal intervals. We saw that (27), the integral representation of the Dirac delta function, lies at the heart of the continuous Fourier transform. The following can be thought of as its discrete analogue

$$\begin{aligned}\sum_{n=0}^{N-1} \exp\left(\frac{2\pi i(k-k')n}{N}\right) &= \frac{1-\exp(2\pi i(k-k'))}{1-\exp(2\pi i(k-k')/N)} = 0, \quad k \neq k' \\ &= N, \quad k = k'\end{aligned} \tag{80}$$

(This is just the sum of a geometric series with a finite number of terms.) Using this we can construct the discrete Fourier transform pair as follows

$$\tilde{a}_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \exp\left(\frac{2\pi ink}{N}\right) a_n; \quad a_n = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \exp\left(\frac{-2\pi ink}{N}\right) \tilde{a}_k \tag{81}$$

We will discuss the relationship between this DFT and continuous Fourier transform shortly. Sometimes the *N* factor is included asymmetrically; this is just a matter of definition and convention, but should be checked up on when using a new piece of software. Given a set of *N* data points, we see that, in the generation of the *N* elements of the corresponding DFT, we will have to perform of the order of *NXN* operations. This implies that the DFT is computationally expensive. However it is possible to reduce the computation involved quite spectacularly.

This speeding up of the DFT depends crucially on the number of data values processed. Let us first require that we analyse an even number of data, i.e. that $N = 2M$. We then have

$$\begin{aligned}\tilde{a}_k(N) &= \sum_{n=0}^{N-1} a_n \exp\left(\frac{2i\pi kn}{N}\right) = \sum_{n=0}^{2M-1} a_n \exp\left(\frac{i\pi kn}{M}\right) \\ &= \sum_{m=0}^{M-1}\left(a_{2m} \exp\left(\frac{\pi ik2m}{M}\right) + a_{2m+1} \exp\left(\frac{\pi ik(2m+1)}{M}\right)\right) \\ &= \sum_{m=0}^{M-1} a_{2m} \exp\left(\frac{2\pi ikm\pi}{M}\right) + \exp\left(\frac{ik\pi}{M}\right)\sum_{m=0}^{M-1} a_{2m+1} \exp\left(\frac{2\pi ikm}{M}\right) \\ &= \tilde{a}_k^e(M) + \exp\left(\frac{ik\pi}{M}\right)\tilde{a}_k^o(M)\end{aligned} \tag{82}$$

Here we have split the DFT of 2*M* points into a combination of two DFTs, each of *M* points derived from either the even or odd order terms in the original sequence (we've also suppressed all the $\sqrt{N}$ s for the moment). That in itself is hopeful. Furthermore each of the $\tilde{a}_k^e(M), \tilde{a}_k^o(M)$ is used twice in determining $\tilde{a}_k(N)$, for *k* less than and greater than *M*:

$$\begin{aligned}\tilde{a}_k(N) &= \tilde{a}_k^e(M) + \exp\left(\frac{ik\pi}{M}\right)\tilde{a}_k^o(M); \quad k \le M \\ &= \tilde{a}_{k'}^e(M) + \exp\left(\frac{i(k'+M)\pi}{M}\right)\tilde{a}_{k'}^o(M); \quad k = M + k' \\ &= \tilde{a}_{k'}^e(M) - \exp\left(\frac{ik'\pi}{M}\right)\tilde{a}_{k'}^o(M) \quad k = M + k'\end{aligned} \tag{83}$$

If M is also an even number, then we can do this again, reducing the computation to that of four DFTs of $M/2 = N/4$ points, of which optimum use is again made in reconstructing the M point DTFs. So, if we take N to be an integral power of 2 we can recur this process down to the point where we have to N 1 point DTFs, which are trivial to evaluate. These are just the original data points, re-ordered as a result of their identification as even and odd order terms in each successive sub DFT. To keep track of this shuffling about is probably easiest to consider a simple example. In an eight-point DFT we have the data points

$$\{a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$$

It is helpful to express the suffices in binary notation as

$$\{a_{000}, a_{001}, a_{010}, a_{011}, a_{100}, a_{101}, a_{110}, a_{111}\} \tag{84}$$

Application of our reduction process regroups these as

$$\{a_0, a_2, a_4, a_6\}, \quad \{a_1, a_3, a_5, a_7\} \text{ or } \{a_{000}, a_{010}, a_{100}, a_{110}\}, \quad \{a_{001}, a_{011}, a_{101}, a_{111}\}$$

Note that in the first set all the final digits in the binary labeling are 0, while in the second they are 1. Carrying on in this way we get

$$\{a_0, a_4\}, \{a_2, a_6\}, \{a_1, a_5\}, \{a_3, a_7\} \text{ or } \{a_{000}, a_{100}\}, \{a_{010}, a_{110}\}, \{a_{001}, a_{101}\}, \{a_{011}, a_{111}\}$$

This re-ordering is based on the second digit being 1 or 0

and

$$\{a_0\}, \{a_4\}, \{a_2\}, \{a_6\}, \{a_1\}, \{a_5\}, \{a_3\}, \{a_7\} \text{ or } \{a_{000}\}, \{a_{100}\}, \{a_{010}\}, \{a_{110}\}, \{a_{001}\}, \{a_{101}\}, \{a_{011}\}, \{a_{111}\} \tag{85}$$

This re-ordering is based on the third digit being 1 or 0

When we compare this final ordering (85) of the terms with that in (84) we see that the binary suffices in the latter are converted to those in the former by bit reversal, as we might expect. This is the general rule for setting up data for FFT. Adjacent pairs are then combined in sums and differences to give 2-point DFTs; these are now combined with phases of 1,i,-1,-i to give 4-point DFTs. As the reduction is run in reverse we are effectively building up the phases in (81) by ever smaller increments. The final step yields the hoped for DFT. So why is it fast? At each level the number of operations required is proportional to N; however only log$N$ iterations of this process are required to generate the DFT from the input data. All told, then, the FFT requires O(Nlog$N$) operations as opposed to the O(NXN) operations of the DFT. When N is reasonably large the difference in computational times is massive. The FFT is definitely clever, so much so that engineers and mathematicians vie to take credit for its discovery. Cooley and Tukey are given much of the glory, 'Numerical Recipes' citing a reduction in computational time for a one million point DFT from two weeks to 30 seconds. Miserable pedants point out that Gauss recorded the basic ideas at the beginning of the 19^{th} century but, as he didn't have a computer to hand to implement it on, his saving in 'CPU time' was from a couple of million years to a lifetime's unremitting toil.

The FFT comes into its own when applied to something like real data; let's apply it to some coherent clutter (simulated using methods we will cover in a later session). This is sampled with a PRF of 1 kHz; its characteristic Doppler frequency is 100Hz.

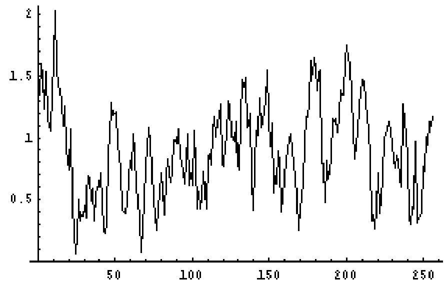


Figure 2: simulated coherent clutter amplitude

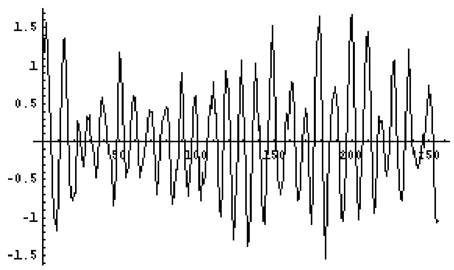


Figure 3: Real part of coherent clutter signal

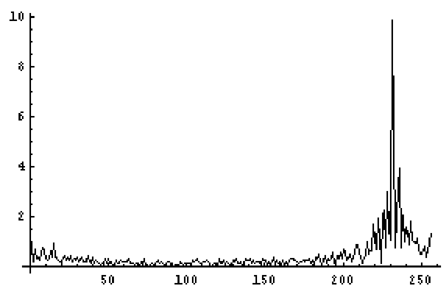


Figure 4: Amplitude of FFT of complex data

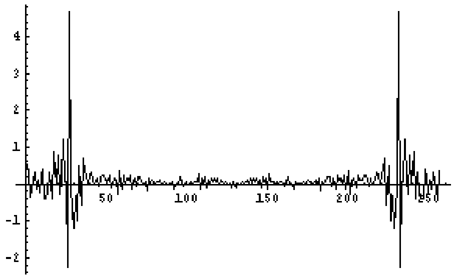


Figure 5: Real part of FFT of real part of the clutter signal

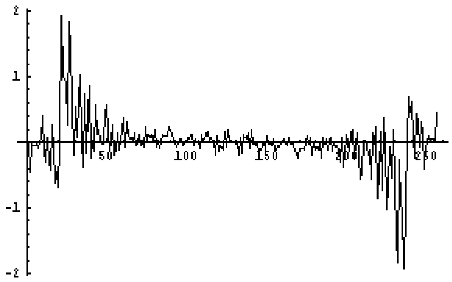


Figure 6 Imaginary part of FFT of real part of clutter signal.

Figure 3 shows the Doppler-derived oscillation that is masked in the amplitude plot. The FFT of the complex signal shown in Figure 4 picks out the Doppler frequency; we can also see the effect of aliasing or wrap around in the region of the zero frequency bin. When we FFT a real function we see the symmetry in the real part of its output (Fig 5), and the anti-symmetry in its imaginary part (Fig 6). Most of the time, one can quite happily use the FFT as a tool, without too much worry about how it actually relates to the Fourier transform. However, there always comes a point where things have to be sorted out: where do all the bits fit in to get the output of the FFT to tie up with something like

$$\int_{-\infty}^{\infty} \exp\left(-t^2\right)\exp(i\omega t)dt = \sqrt{\pi}\exp\left(-\omega^2/4\right)? \tag{86}$$

Before we set about this we have one more thing to say about the FFT: the way it distributes its frequency components among its bins. For $0 \le k < N/2$ the k^{th} bin is associated with a ‘frequency’ of $2\pi k/N$; for $k > N/2$ we note that

$$\exp\left(i\frac{2\pi}{N}\left(\frac{N}{2}+q\right)\right) = \exp\left(i\frac{2\pi}{N}\left(N-\left(\frac{N}{2}-q\right)\right)\right) = \exp\left(-i\frac{2\pi}{N}\left(\frac{N}{2}-q\right)\right); \quad q < N/2. \tag{87}$$

Consequently the negative frequency components end up, in reverse order, in the second half of the output array. This observation relates the appearance of Figures 5 & 6 to the analytic result.

To make direct contact between the FFT output and the Fourier transform we first make recourse to the sampling theorem. This tells us that

$$\tilde{f}(\omega) = \Delta\sum_{-\infty}^{\infty} f(n\Delta)\exp(i\omega n\Delta) \equiv \Delta\sum_{-\infty}^{\infty} f((n+1/2)\Delta)\exp(i(n+1/2)\Delta\omega)$$
$$\Delta < \frac{\pi}{\omega_B} \tag{88}$$

This little shift doesn't make any difference to the answer (it doesn't matter where you take your origin when you integrate, as long as you fix up the integrand and the limits of integration) We have done this because the FFT operates on an even number of data points. The function *f* is taken to be localised; we truncate the sum in (88) to have *N* terms

$$\tilde{f}(\omega) = \Delta \sum_{n=-N/2}^{N/2} f((n+1/2)\Delta)\exp(i(n+1/2)\Delta\omega) = \Delta\sum_{n=0}^{N-1} f((n-(N-1)/2)\Delta)\exp(i(n-(N-1)/2)\omega) \tag{89}$$

If we now make the identification

$$\omega = \frac{2\pi k}{N\Delta} \tag{90}$$

we can relate the Fourier transform of *f*, sampled at frequencies that are integer multiples of $2\pi/N\Delta$, to the output of a FFT:

$$\tilde{f}\left(\frac{2\pi k}{N\Delta}\right) = \Delta(-1)^k \exp\left(\frac{i\pi k}{N}\right)\sum_{n=0}^{N-1} f((n-(N-1)/2)\Delta)\exp\left(\frac{2\pi ikn}{N}\right). \tag{91}$$

(You might like to check that the peak in Figure 4 does indeed correspond to a Doppler shift of 100Hz.) This prescription allows us to tie down the connection between the output of a FFT (remembering to take account of where the $\sqrt{N}$ crops up in the software you are using) and the pencil wielder's efforts, should we need to. Untold boffin hours have been dissipated in pursuit of πs, *N*s and Δs, so it might be a good idea to check all this through on something fairly benign like (86).

Radar imaging and Fourier analysis

As a prelude to our discussion of the imaging process we first recall the Fourier integral theorem, which identifies the transform pair (21) This demonstrates how a function $f(x)$ is decomposed into a set of Fourier components $\tilde{f}(k)$. These can in turn be combined to re-construct the 'target'. Taken together (21) provide us with an integral representation of the Dirac delta function (23) which satisfies the relations (22) In this simple case where we represent, or image, *f* by Fourier reconstruction, we can think of the delta function as a point spread function which, when convolved with the target function, produces its image.

If all the Fourier components of *f* are not available, any attempt at Fourier reconstruction will result in a degraded 'image' $\hat{f}$ of *f*. This degradation can be represented by introducing a weighting in *k* space into the second of equations (21); *S* takes small values in the regions of *k* space that are not accessible to the imperfect imaging process. Thus we write

$$\hat{f}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-ikx)S(k)\tilde{f}(k)dk \tag{92}$$

We see that the point-spread function, which is convolved with the target to produce the degraded image, is itself the inverse Fourier transform of the *k* space weighting

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(x')\Delta(x - x')dx'$$
$$\Delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-ikx)S(k)dk \qquad (93)$$

This simple Fourier representation of the imaging process is very useful, both conceptually and in the calculation of radar imaging performance and its degradation. Thus, for example, this representation of the point spread function has been applied quite extensively in the study of the effects of correlated phase noise on radar imaging, in the particular context of motion compensation, and various SAR, ISAR modes. However, standard discussions of radar detection and imaging do not lend themselves to an immediate interpretation in terms of these concepts.

The fundamental operation in radar detection is the transmission of a pulse, of some duration τ, which travels to the target, is reflected and travels back to the receiver. Its time of flight T along this two-way path is related directly to the range *R* to the target by

$$T = 2R/c\,; \qquad (94)$$

here c is the velocity of light. The accuracy with which this range can be resolved increases as the pulse is shortened; the range over which a pulse can be transmitted depends on its energy, which in turn becomes smaller as the pulse becomes shorter. These contradictory requirements of range and accuracy can be reconciled by transmitting a time varying pulse. The returned pulse is in effect a copy of the transmitted pulse delayed by a time *T*, which may be corrupted, for example, an additive noise process. By correlating this received pulse with (the complex conjugate of) the transmitted pulse one achieves several things. This procedure maximises the signal to noise ratio in its output which relates directly to a sufficient statistic for the likelihood ratio based detection of a known signal corrupted by additive Gaussian noise . These issues will be discussed in a later session. Furthermore, the time varying pulse is transmitted over a reasonable length of time and is able to carry a substantial amount of energy over a long range. The correlation processes nonetheless focuses down (‘compresses’) the pulse to achieve a much better range accuracy than could be obtained from a simple, uniform pulse of the same duration. This process of pulse compression is very widely used in high-resolution radar systems.

At first sight pulse compression processing appears to make little contact with the Fourier representation of the point spread function introduced at the start of this section. (It should be noted that FT methods are often used to effect the correlation step in pulse compression in digital systems; this is a matter of computational expediency, exploiting the extremely rapid fast Fourier transform (FFT) algorithm.) If, however, we assume that the *k*-space weighting function *S* can be chosen to be real and positive (as is the case, for example, for ‘top-hat’, Gaussian and Hanning weightings) then we can write

$$S(k) = \left|\tilde{\sigma}(k)\right|^2 \qquad (95)$$

where

$$\tilde{\sigma}(k) = \int_{-\infty}^{\infty} \exp(ikx)\sigma(x)dx \tag{96}$$

and

$$\tilde{\sigma}(k)^* = \int_{-\infty}^{\infty} \exp(-ikx)\sigma(x)^* \, dx = \int_{-\infty}^{\infty} \exp(ikx)\sigma(-x)^* \, dx \,. \tag{97}$$

When these results are introduced into the expression for the point spread function, and the convolution theorem for Fourier transforms is exploited we find that

$$\Delta(x) = \int_{-\infty}^{\infty} \sigma(x - x')\sigma(-x')^* \, dx' = \int_{-\infty}^{\infty} \sigma(x + x')\sigma(x')^* \, dx' \tag{98}$$

Thus the *k* space representation of the point-spread function relates directly to the output of the correlation process characteristic of pulse compression. Conversely the associated *k* space weighting is identified with the Fourier transform of the correlation of the transmitted pulse with a delayed copy of itself.

In this discussion we have represented the imaging process in configuration space. Discussions of pulse compression are invariable couched in the time domain; the link between time and displacement is made through the identification of the former as a time of flight (94). Standard discussions of SAR processing address the problem of azimuthal compression in the time domain; nonetheless considerable insight is achieved from an equivalent *k* space formulation similar in spirit to that we have adopted here. So, for the present we shall consider both perspectives on the problem. To illustrate some of the salient features of the pulse compression process we consider the finite duration, chirped pulse, which we represent as follows

$$\begin{aligned} \sigma(x) &= \exp\left(i\alpha x^2\right), \quad |x| \le Q \\ &= 0, \quad |x| \ge Q \end{aligned} \tag{99}$$

From this we obtain the Fourier transform

$$\hat{\sigma}(k) = \int_{-Q}^{Q} \exp(ikx)\exp\left(i\alpha x^2\right)dx = \exp\left(-i\,k^2/4\alpha\right) \int_{-Q+k/2\alpha}^{Q+k/2\alpha} \exp\left(it^2\right)dt \tag{100}$$

The integral occurring here can be expressed in terms of Fresnel integrals, which are widely tabulated. In the next session we will see how *Mathematica* allows us to handle these possibly unfamiliar functions.

$$\begin{aligned} \mathrm{FresnelC}(x) &= \int_0^x \cos\left(\frac{\pi t^2}{2}\right)dt \\ \mathrm{FresnelS}(x) &= \int_0^x \sin\left(\frac{\pi t^2}{2}\right)dt \end{aligned} \tag{101}$$

Thus we have

$$\hat{\phi}(k)=\sqrt{\frac{\pi}{2\alpha}}\exp\left(-ik^2/4\alpha\right)\left\{\mathrm{FresnelC}\left(\frac{k+2\alpha Q}{\sqrt{2\pi\alpha}}\right)-\mathrm{FresnelC}\left(\frac{k-2\alpha Q}{\sqrt{2\pi\alpha}}\right)+i\left(\mathrm{FresnelS}\left(\frac{k+2\alpha Q}{\sqrt{2\pi\alpha}}\right)-\mathrm{FresnelS}\left(\frac{k-2\alpha Q}{\sqrt{2\pi\alpha}}\right)\right)\right\}$$

(102)

whose complex conjugate is written similarly. This leads us to a *k* space weighting function *S* that can be written as

$$\begin{aligned}S(k)&=\hat{\sigma}(k)\hat{\sigma}(k)^*\\&=\frac{\pi}{2\alpha}\left(\left(\mathrm{FresnelC}\left(\frac{k+2\alpha Q}{\sqrt{2\pi\alpha}}\right)-\mathrm{FresnelC}\left(\frac{k-2\alpha Q}{\sqrt{2\pi\alpha}}\right)\right)^2+\left(\mathrm{FresnelS}\left(\frac{k+2\alpha Q}{\sqrt{2\pi\alpha}}\right)-\mathrm{FresnelS}\left(\frac{k-2\alpha Q}{\sqrt{2\pi\alpha}}\right)\right)^2\right)\end{aligned}$$

(103)

This *k* space weighting function displays a great deal of fine structure, as can be seen from the plot shown in Figure 7. In the limit of large α *S* tends to a top-hat limiting form

$$\begin{aligned}S(k)&=\frac{\pi}{\alpha}\quad |k|\leq 2\alpha Q\\&=0\quad |k|>2\alpha Q\end{aligned}\qquad(104)$$

This approximation can also be obtained directly from (100) by a stationary phase argument.

While we might attempt to evaluate the corresponding point spread function by Fourier inversion of this expression it is much more convenient to calculate it directly in configuration space. Thus we form

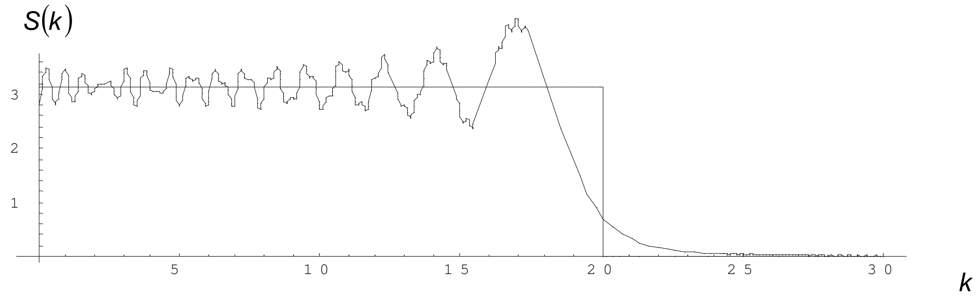


Figure 7. The k-space weighting function associated with a chrped pulse, along with the 'top-hat' limiting form. The fine structure present in (16) is evident; we have chosen parameter values $\alpha=1,\quad Q=10$

$$\begin{aligned}\Delta(x) &= \int_{-\infty}^{\infty} \sigma(x+y)\sigma(y)^* \, dy \\ &= \exp\left(i\alpha x^2\right) \int_{-Q}^{Q-x} \exp(2i\alpha xy)dy, \quad 2Q \geq x \geq 0 \\ &= \exp\left(i\alpha x^2\right) \int_{-Q-x}^{Q} \exp(2i\alpha xy)dy, \quad -2Q \leq x \leq 0 \\ &= \frac{\sin(\alpha|x|(2Q-|x|))}{\alpha|x|}, \quad -2Q \leq x \leq 2Q \\ &= 0; \quad |x| > 2Q\end{aligned} \tag{105}$$

This can be compared with the limiting form obtained from the top hat weighting

$$\Delta(x) = \frac{1}{2\alpha} \int_{-2Q\alpha}^{2Q\alpha} \exp(ikx)dk = \frac{\sin(2Qx\alpha)}{\alpha x} \tag{106}$$

On examining (105) we note that the first zero in this point-spread function occurs at

$$|x| = Q - Q\sqrt{1 - \pi/2\alpha Q^2} \approx \frac{\pi}{2Q\alpha} \tag{107}$$

the approximate equality holding when αQ^2 is large. This important parameter is a measure of the product of the pulse duration ('time') and the range of frequencies it contains ('bandwidth'). Thus we see that, while the uncompressed pulse is of length $2Q$, compression reduces this to $\pi/Q\alpha$ and high resolution in range is achieved.

Exercises

The application of Fourier and related methods requires a facility in both formal manipulation and the use of computer software. The following exercises should give some practice in both. (For the numerical/qualitative stuff, use whatever software you like.)

1 Consider the trigonometric series

$$S_N(x) = \sum_{n=1}^{N} \frac{\sin(nx)}{n}.$$

What function is represented by the Fourier series $S_\infty(x)$?

Sketch out the behaviour of $S_N(x)$ (with large but not infinite *N*) over the range $-2\pi < x < 2\pi$.

2 Represent the function $f(x) = x^2$ as a Fourier series over the range $-\pi \le x \le \pi$. Use the series you obtain to evaluate the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Construct another Fourier series that you can use to evaluate $\sum_{n=1}^{\infty} \frac{1}{n^4}$. Confirm your result by constructing the analogue of Parseval's theorem (36) (discussed in the session for Fourier transforms) applicable to Fourier series and applying it to your representation of x^2.

3 Evaluate the Fourier transforms of the following functions

$x\exp(-a|x|)$

$x\exp(-ax^2)$

$x^2\exp(-ax^2)$

$\sin(bx)\exp(-ax^2)$

$\cos(x)$

$\cos(x^2)$

$1/(a^2+x^2)$

$\sin(bx)/(a^2+x^2)$

$f(x) = 1-|x|; \quad -1 < x < 1 \quad = 0,$ otherwise

$(1-x^2)^{n-1} \quad -1 < x < 1 \quad = 0,$ otherwise

Evaluate $\int_0^\infty dx\cos(ax)\exp(-bx^2)$, think about $\int_0^\infty dx\sin(ax)\exp(-bx^2)$

4 Consider the top-hat function

$$\begin{aligned} H(x) &= \frac{1}{a}; |x| \le \frac{a}{2} \\ &= 0; |x| > \frac{a}{2} \end{aligned}$$

Evaluate the convolution of two top hat functions, directly and via their Fourier transforms. If you wish you can also attempt this problem using a FFT or related numerical algorithm. Using any and every means at your disposal, investigate the three, four and higher *n*- fold convolutions of the top hat function. (A can of Fanta for anyone who can produce a general formula.) What can you say about the *n*-fold convolution when *n* gets big?

5 Take some function whose Fourier transform you can evaluate analytically. Match up this result with the output of a suitably applied FFT algorithm. Investigate effects of sub-Nyquist sampling, aliasing, sinc interpolation etc. for a variety of functions, some to which the concept of a bandwidth ω_B might reasonably be applied and others to which it might not. (Lots of graphs expected here.)

6 Confirm the result given in (103); reproduce Figure 7 for a variety of Q, α values, and derive the limiting form (104) by stationary phase analysis, as suggested in the session notes.