All you ever wanted to know about flashy sums etc.

In this session we complete our preparation for the analysis and modeling of radar performance. To do this we de-mystify those rather arcane bits and pieces that are needed in addition to the more conventional, 'maths for scientists', material covered in the previous talks. There are several reasons for doing this. Much of the theory underlying radar was developed some fifty or more years ago, when the mathematical education of scientists and engineers was different from that of today. As computers were at that time in their infancy, a greater emphasis was placed on analytical methods than is now the case. Conversely, much less was done numerically and through simulation. When we look at, for example, Middleton's 'Introduction to Statistical Communication theory' or the classic papers gathered together in Haykin's 'Detection and Estimation', we are confronted by an impenetrable mass of formalism, incorporating Bessel functions, elliptic integrals, poly-logarithms and the like. At first sight these unfamiliar monstrosities present a barrier, preventing our access to the valuable insights that this work undoubtedly contains. Much of the journal material published subsequently, which builds on this classic work, suffers from the same problem. If, however, we are able to surmount such an obstacle, this literature might well turn out to be invaluable. While the development of computers has left us ill-prepared to deal with such occult formal work, it has, paradoxically, made it potentially even more useful. The last decade or so has seen the emergence of software that renders this rather forbidding mathematics much more accessible. The hard-won closed form expressions of yore were sometimes almost an end in themselves; their transmutation into useful numerical results required a significant investment in rather specialised programming. This, perhaps, accounts for the vast tabulations of graphical and numerical results found in Meyer and Mayer's 'Radar Target Detection' and in Abramowitz and Stegun's 'Handbook of Mathematical Functions'. Nowadays efficient algorithms, able to evaluate these functions accurately over a wide range of parameters, are available, and are incorporated into several commercial software packages. Of these, *Mathematica* is perhaps best able to carry out formal manipulations, and to capitalise on an encyclopaedic built-in prior knowledge of the properties of a whole host of functions. The *Mathematica* website also provides access, via functions.wolfram.com and Eric Weinstein's World of Mathematics, to an enormous, if not plethoric, abundance of results. Today we identify those 'special functions' that crop up most regularly in radar performance work, and see that a few relatively simple tricks, based on no more than a bit of algebra and calculus, allow us to manipulate them quite effectively. We also look at *Mathematica* in action, reducing the labour involved in the evaluation and deployment of these functions to a manageable level. And that, effectively, is all you should ever need to know about flashy sums.

Just as in the case of the elementary transcendental functions, there are several avenues down which these special functions can be pursued. Thus we could start from a power series expansion, much as we might introduce the logarithmic function through

$$\log(1+x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}(-1)^n \qquad (1)$$

Alternatively we might start from an integral representation of the function

$$\log(1+x) = \int_0^x \frac{dx'}{(1+x')} \, . \qquad (2)$$

or from the differential equation it satisfies

$$\frac{d\log(1+x)}{dx} = \frac{1}{1+x} \, . \qquad (3)$$

In this simple example these starting points are so closely related as to be virtually indistinguishable. When we consider the special functions, these several routes, from more widely separated starting points, complement and illuminate each other and are, ultimately, mutually consistent. Any of these approaches can be chosen, for convenience and prior familiarity. We will investigate all of them in today's session. Rather than explore the whole gamut of special functions, we confine our attention to a limited, but practically useful, selection. The gamma function and its close relatives, the hypergeometric function and its generalised and confluent forms, some orthogonal polynomials (which you may well have met when you studied quantum mechanics) and our old friends, the Bessel functions. As will become apparent in future sessions, this choice is based on these functions' utility in modeling radar performance.

Gamma and related functions.

As a preliminary to a discussion of series expansions of special functions we recall the binomial expansion, for positive integer exponents:

$$(1+x)^n = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} x^r \tag{4}$$

The extension of the binomial theorem to a negative, integer exponent is effected by

$$\begin{aligned}(1+x)^{-n} &= \sum_{r=0}^{\infty} \frac{-n(-n-1)\cdots(-n-r+1)}{r!} x^r \\ &= \sum_{r=0}^{\infty} \frac{(-1)^n n(n+1)\cdots(n+r-1)}{r!} x^r \\ &= \sum_{r=0}^{\infty} \frac{(-1)^n (n+r-1)!}{r!(n-1)!} x^r\end{aligned} \tag{5}$$

The coefficients in these two power series have been expressed in terms of the factorial function $m!$; if we could identify the non-integer generalisation of this factorial function we could express the binomial theorem, with a general exponent, in much the same way. Thus we would have

$$\begin{aligned}(1+x)^{-\alpha} &= \sum_{r=0}^{\infty} \frac{(-1)^r \alpha(\alpha+1)\cdots(\alpha+r-1)}{r!} x^r \\ &= \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(\alpha+r)}{r!\,\Gamma(\alpha)} x^r\end{aligned} \tag{6}$$

where

$$\Gamma(z+1) = z\Gamma(z); \qquad \Gamma(n+1) = n!\,. \tag{7}$$

So how do we interpolate between the integer values of $m!$?. We could, of course, do it any way we liked, with all sorts of wiggling between the integer arguments. If we require the interpolating function to be convex and analytic (for positive real part of z) then we are led inescapably to the gamma function, which we have met before:

$$\Gamma(z) = \int_0^\infty t^{z-1} \exp(-t) dt \,. \tag{8}$$

This interpolation was devised by Euler; an alternative representation of the interpolating function, derived by Gauss, is

$$\Gamma(z) = \lim_{n \to \infty} \Gamma(z,n); \quad \Gamma(z,n) = \frac{n!\, n^z}{z(z+1)\cdots(z+n)} \tag{9}$$

and is occasionally quite useful (the relationship between these is investigated in an exercise). As the coefficients in series expansions of special functions can often be expressed, and then manipulated, in terms of the gamma function we will look at its properties in a bit more detail. These also underpin the use of the gamma distribution in clutter modeling, so they will be immediately useful in later sessions. In fact we have already done quite a bit of the required work in previous sessions, disguised as examples and exercises. Thus we recall how the fundamental recurrence relation $\Gamma(z+1) = z\Gamma(z)$ is established by integrating by parts. The leading term in the asymptotic expansion of the gamma function for large positive argument has also been established in our discussion of Laplace's method

$$\Gamma(z) \sim \exp(-z) z^{z-1/2} \sqrt{2\pi} \,. \tag{10}$$

It's interesting to note that the next few terms in the expansion

$$\Gamma(z) \sim \exp(-z) z^{z-1/2} \sqrt{2\pi} \left[1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - \frac{571}{2488320z^4} \cdots \right] \tag{11}$$

gives us a 'large *z*' expansion that is accurate to better than one part in 1,000 when *z* is equal to unity. We also saw in the exercises for the first session that, when *n* is an integer,

$$\pi^{1/2} (2n)! = 2^{2n} n!\, \Gamma(n + 1/2) \,. \tag{12}$$

Before we go on to extend these formal properties of the gamma function, we should say a bit about its behaviour for various values of its argument. We see from the definition in terms of the integral that the gamma function is positive and greater than zero for real *z* greater than 0. What do we do when $z \le 0$?. If *z* is not a negative integer we can make repeated use of the fundamental recurrence relation to evaluate $\Gamma(z)$ in terms of a gamma function with an argument with a real part greater than zero. To investigate the behaviour of $\Gamma(z)$ for *z* =-*n* we first 'split off' the portion of the integrand in the vicinity of the origin:

$$\Gamma(z) = \Psi(z) + \Phi(z); \quad \Psi(z) = \int_0^1 t^{z-1} \exp(-t) dt \qquad \Phi(z) = \int_1^\infty t^{z-1} \exp(-t) dt \tag{13}$$

We see that $\Phi(z)$ is well behaved for all *z*; can we express $\Psi(z)$, in a form, based initially on the integral representation, that is valid for $z \le 0$? Expand the exponential and integrate term by term:

$$\Psi(z) = \int_0^1 t^{z-1} \sum_{n=0}^\infty \frac{(-t)^n}{n!} dt = \sum_{n=0}^\infty \frac{(-1)^n}{(z+n)\, n!} \tag{14}$$

This representation of $\Psi(z)$ coincides with the integral form for *z* greater than zero, but is still analytic in the $z \le 0$ region, except at the negative integer values of *z* where it reveals simple poles (at $z = -n$). It now remains to flesh out and extend the results we have encountered already in the exercises. This is frequently no more than a bit of low cunning. For example, we can show that

$$\int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{15}$$

To prove this we first make the change in variable $t = s/s+1$; the integral becomes

$$\int_0^\infty \frac{s^{a-1}}{(1+s)^{a+b}}ds \tag{16}$$

Now introduce the integral representation

$$\frac{1}{(1+s)^{a+b}} = \frac{1}{\Gamma(a+b)}\int_0^\infty \exp(-(1+s)p)p^{a+b-1} \tag{18}$$

and invert the order of integration; the required result follows immediately:

$$\begin{aligned}\int_0^\infty \frac{s^{a-1}}{(1+s)^{a+b}}ds &= \frac{1}{\Gamma(a+b)}\int_0^\infty dp\exp(-p)p^{a+b-1}\int_0^\infty dss^{a-1}\exp(-ps)\\ &= \frac{\Gamma(a)}{\Gamma(a+b)}\int_0^\infty dp\exp(-p)p^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\end{aligned} \tag{19}$$

Two useful properties of the gamma function are the duplication formula

$$\pi^{1/2}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2) \tag{20}$$

and the reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \tag{21}$$

The former can be proved using elementary calculus; one way of doing this is suggested in the exercises. The latter is more problematic when using only elementary methods. If one can show that

$$\int_0^\infty \frac{t^{z-1}dt}{1+t} = \frac{\pi}{\sin \pi z}, \tag{22}$$

all is well. Integrations of this type are best attacked in the complex plane, exploiting Cauchy's theorem (See exercise 3, session 3). In fact this very integral was evaluated by Cauchy, using his private methods; he then proceeded to taunt his fellow sum bashers with it in a most ignoble fashion. If you know how to do this (that is, contour integration), you're away; if you don't, you are in the same boat as Cauchy's mates. (All serious noodling about with contour integrals has been excised from this course.) You can also use the Gauss product representation to extract this result, as Euler did himself; to do this, however, you need to know how to represent sines and cosines as infinite products. So take it, or leave it, on trust

The analysis of the performance of the mean of the log intensity as a discriminant between different clutter textures in a SAR image, for example, focuses attention on quantities like

$$\int_0^\infty (\log(x))^n x^{\nu-1}\exp(-bx)dx \tag{23}$$

Low cunning suggests that we might evaluate this by differentiating under the integral sign:

$$\int_0^\infty (\log(x))^n x^{\nu-1}\exp(-bx)dx = \frac{\partial^n}{\partial\nu^n}\int_0^\infty x^{\nu-1}\exp(-bx)dx = \frac{\partial^n}{\partial\nu^n}\frac{\Gamma(\nu)}{b^\nu}. \tag{24}$$

The bit with the *b* in is alright; how do we attack the derivatives of the gamma function? Gauss's product representation comes to our aid:

$$\begin{aligned}
&\log((z,n)) = \log(n!) + z\log n - \sum_{r=0}^{n}\log(z+r)\\
&\frac{d\log(\Gamma(z,n))}{dz} = \log n - \frac{1}{z} - \sum_{r=1}^{n}\frac{1}{(z+r)}\\
&\frac{d\log(\Gamma(z))}{dz} = -\frac{1}{z} - \sum_{r=1}^{\infty}\left[\frac{1}{(z+r)} - \frac{1}{r}\right] + \lim_{n\to\infty}\left(\log n - \sum_{r=1}^{n}\frac{1}{r}\right)\\
&\qquad = -\gamma - \frac{1}{z} + \sum_{r=1}^{\infty}\left[\frac{1}{r} - \frac{1}{(z+r)}\right]\\
&\frac{d^2\log(\Gamma(z))}{dz^2} = \frac{1}{z^2} + \sum_{r=1}^{\infty}\frac{1}{(z+r)^2}
\end{aligned} \tag{25}$$

The constant $\gamma \approx 0.5772$ is sufficiently famous to have a name (perhaps not surprisingly, Euler's) attached to it. The first and second derivatives of the logarithm of the gamma function are frequently referred to as the digamma and trigamma functions.

Series expansions of some special functions

We can now begin to look at the power series representations of some special functions; before we do, though, we will introduce a bit of notation, the Pochhammer symbol. This is defined by

$$\begin{aligned}
(\alpha)_n &= \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\\
&= \alpha(\alpha+1)\cdots(\alpha+n-1)
\end{aligned} \tag{26}$$

and has, as special cases, the values

$$(1)_n = n! \quad (-n)_r = (-1)^r \frac{n!}{(n-r)!}. \tag{27}$$

Using this notation we can write the familiar binomial theorem in the form

$$\frac{1}{(1-x)^\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} x^n. \tag{28}$$

The hypergeometric series is characterised by three parameters (α, β, γ) in addition to the argument z. The standard notation for this function is

$${}_2F_1(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} z^n \tag{29}$$

We require that $|z|<1$ to ensure absolute convergence of this series. Obviously ${}_2F_1(\alpha,\beta;\gamma;z) = {}_2F_1(\beta,\alpha;\gamma;z)$. The confluent hypergeometric series takes the form

$${}_1F_1(\alpha;\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n n!} z^n \tag{30}$$

and is absolutely convergent for all values of z. (It is illuminating to compare these series expansions with that of the exponential function and the binomial theorem.) A generalised hypergeometric function ${}_PF_Q$ can also be defined

$${}_PF_Q(\alpha_1,\cdots,\alpha_P;\beta_1,\cdots,\beta_Q;z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n\cdots(\alpha_P)_n}{(\beta_1)_n\cdots(\beta_Q)_n n!} z^n \tag{31}$$

whose convergence properties depend on the relative values of P and Q. Note that the series terminates as a polynomial if any of the $\alpha_1,\cdots,\alpha_P$ is a negative integer. These series have a sufficiently general structure to accommodate many of the special functions as well as the familiar exponentials, trigs etc. A less than exhaustive compilation of elementary functions expressed in hypergeometric form follows; perhaps you would like to verify some of them. The gamma function duplication formula comes in very useful when you are doing this.

$${}_0F_0(z) = \exp(z)$$

$${}_1F_0(\alpha;z) = \frac{1}{(1-z)^\alpha}$$

$${}_0F_1\left(1/2;-z^2/4\right) = \cos(z)$$

$${}_2F_1(1,1;2;z) = -\frac{\log(1-z)}{z}$$

$${}_2F_1\left(1/2,1;3/2;z^2\right) = \frac{1}{2z}\log\left(\frac{1+z}{1-z}\right)$$

$${}_2F_1\left(1/2,1/2;3/2;z^2\right) = \frac{\sin^{-1}(z)}{z}$$

$${}_2F_1\left(a,a+1/2;1/2;z^2\right) = \frac{\left((1+z)^{-2a} - (1-z)^{-2a}\right)}{2}$$

$${}_2F_1(a,a+1/2;2a;z) = 2^{2a-1}(1-z)^{-1/2}\left[1+(1-z)^{1/2}\right]^{1-2a} \tag{32}$$

$${}_2F_1\left(-a,a;1/2;\sin^2 z\right) = \cos(2az)$$

$${}_2F_1\left(1-a,a;3/2;\sin^2 z\right) = \frac{\sin((2a-1)z)}{(2a-1)\sin(z)}$$

Many of the properties of the hypergeometric functions can be derived merely by manipulating these series. This is very straightforward in principle (massage things around and equate coefficients of powers of *z*) but can be a bit messy in practice. A couple of examples:

$$\frac{d}{dz}\,{}_2F_1(a,b;c;z)=\frac{ab}{c}\,{}_2F_1(a+1,b+1;c+1;z)$$

$${}_2F_1(a,b;c;z)=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c)_r\,r!}z^r; \tag{33}$$

$$\frac{d}{dz}\,{}_2F_1(a,b;c;z)=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c)_r\,r!}rz^{r-1}=\frac{ab}{c}\sum_{r=0}^{\infty}\frac{(a+1)_{r-1}(b+1)_{r-1}}{(c+1)_{r-1}(r-1)!}z^{r-1}$$

$$\begin{aligned}\left(\frac{d}{dz}\right)^n\left(z^{n+a-1}\,{}_2F_1(a,b;c;z)\right)&=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c)_r\,r!}\left(\frac{d}{dz}\right)^n z^{n+a+r-1}\\&=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c)_r\,r!}\frac{\Gamma(a+n+r)}{\Gamma(a+r)}z^{a+r-1}\\&=\frac{\Gamma(a+n)}{\Gamma(a)}\sum_{r=0}^{\infty}\frac{(a+n)_r(b)_r}{(c)_r\,r!}z^{a+r-1}\\&=(a)_n z^{a-1}\,{}_2F_1(a+n,b;c;z)\end{aligned} \tag{34}$$

A whole host of recurrence relations between hypergeometric functions whose parameters differ by integer values can be derived in this way; these are tabulated in detail in Abramowitz and Stegun 'A Handbook of Mathematical Functions' Section 15.2. For an example of their use in the solution of a practical problem see Tough, Blacknell and Quegan 'A statistical description of polarimetric and interferometric synthetic aperture radar data.' Proc.Roy.Soc. **A449**, 567, 1995. In particular it can be shown by direct substitution of the series that the hypergeometric function ${}_2F_1(a,b;c;z)$ satisfies the differential equation

$$z(1-z)\frac{d^2y}{dz^2}+\left(c-(a+b+1)z\right)\frac{dy}{dz}-aby=0 \tag{35}$$

The confluent hypergeometric function, whose series representation is

$${}_1F_1(a;b;z)=\sum_{r=0}^{\infty}\frac{(a)_r}{(b)_r\,r!}z^r \tag{36}$$

obeys a range of differential and recurrence relations, that can be derived in the same way,(See Abramowitz and Stegun, Section 13.4) and satisfies the differential equation

$$z\frac{d^2y}{dz^2}+(b-z)\frac{dy}{dz}-ay=0\,. \tag{37}$$

The Bessel function *J* can be written in a power series form, premultiplied by a surd:

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{r=0}^{\infty} \frac{\left(-z^2/4\right)^r}{r!\,\Gamma(\nu + r + 1)} = \left(\frac{z}{2}\right)^\nu \frac{{}_0F_1\left(\nu + 1; -z^2/4\right)}{\Gamma(\nu + 1)} \tag{38}$$

This series form can again be used to demonstrate the recurrence relations

$$\begin{aligned} J_{\nu-1}(z) + J_{\nu+1}(z) &= \frac{2\nu}{z} J_\nu(z) \\ J_{\nu-1}(z) - J_{\nu+1}(z) &= 2\frac{dJ_\nu(z)}{dz} \\ \frac{\nu}{z} J_\nu(z) + \frac{dJ_\nu(z)}{dz} &= J_{\nu-1}(z) \\ \frac{\nu}{z} J_\nu(z) - \frac{dJ_\nu(z)}{dz} &= J_{\nu+1}(z) \end{aligned} \tag{39}$$

from which Bessel's equation (satisfied by his functions) emerges:

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) y = 0\,. \tag{40}$$

The route via the series representation of the Bessel function, through some intermediate process to another series that we can identify leads to lots of Bessel function identities. For example we consider the integral

$$\begin{aligned} \int_0^\infty \exp(-at) J_\nu(bt) t^{\mu-1} dt &= \left(\frac{b}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{\left(-b^2/4\right)}{n!\,\Gamma(n + \nu + 1)} \int_0^\infty \exp(-at) t^{\mu+\nu+2n-1} dt \\ &= \left(\frac{b}{2}\right)^\nu \frac{1}{a^{\mu+\nu}} \sum_{n=0}^{\infty} \frac{\left(-b^2/4a^2\right)}{n!\,\Gamma(n + \nu + 1)} \Gamma(\mu + \nu + 2n) \end{aligned} \tag{41}$$

We re-arrange this by applying the duplication formula to the gamma function in the numerator of the summand. This gives us

$$\begin{aligned} &\left(\frac{b}{2}\right)^\nu \frac{2^{\mu+\nu-1}}{a^{\mu+\nu}\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\left(-b^2/a^2\right)^n}{n!\,\Gamma(n + \nu + 1)} \Gamma\left(n + (\mu + \nu)/2\right) \Gamma\left(n + (\mu + \nu + 1)/2\right) \\ &= \frac{b^\nu 2^{\mu-1} \Gamma\left((\mu + \nu)/2\right) \Gamma\left((\mu + \nu + 1)/2\right)}{a^{\mu+\nu} \Gamma(\nu + 1)\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\left(-b^2/a^2\right)^n}{n!(\nu + 1)_n} \left((\mu + \nu)/2\right)_n \left((\mu + \nu + 1)/2\right)_n \\ &= \frac{b^\nu \Gamma(\mu + \nu)}{2^\nu \Gamma(\nu + 1) a^{\mu+\nu}} \, {}_2F_1\left((\mu + \nu)/2, (\mu + \nu + 1)/2; \nu + 1; -b^2/a^2\right) \end{aligned} \tag{42}$$

Specialising the values of μ, ν to those for which the hypergeometric function can be expressed in terms of elementary functions allows a whole set of results to be extracted from this general formula. However impressive, or pointless, this result looks, it is arrived at by a set of simple steps that are not that hard. Special function manipulations that arise in practice are usually no more difficult than this example.

We will also come across modified Bessel functions (essentially like the *J*s, but with an imaginary argument) that satisfy the differential equation

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} - \left(1 + \frac{\nu^2}{z^2}\right)y = 0 \tag{43}$$

and have the power series expansion

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{r=0}^{\infty} \frac{\left(z^2/4\right)^r}{r!\,\Gamma(\nu + r + 1)} = \left(\frac{z}{2}\right)^\nu \frac{{}_0F_1\left(\nu + 1; z^2/4\right)}{\Gamma(\nu + 1)}; \tag{44}$$

perhaps you would like to look at the recurrence relations satisfied by these fellows.

As we have already seen many elementary functions can be expressed as hypergeometric series. The same is true of many of the more recondite bits and pieces that crop up in radar performance assessment, and elsewhere in physics and engineering. Thus the error function can be written as

$$\operatorname{erf}(z) = \frac{2z}{\sqrt{\pi}}\,{}_1F_1\left(1/2; 3/2; -z^2\right), \tag{45}$$

the Laguerre and Hermite polynomials (familiar from quantum mechanics courses, if from nowhere else) can be identified as terminating confluent hypergeometric series:

$$\begin{aligned} L_n^{(\alpha)}(z) &= \frac{(\alpha + 1)_n}{n!}\,{}_1F_1(-n; \alpha + 1; z) \\ H_{2n}(z) &= (-1)^n 2^{2n} n!\, L_n^{(-1/2)}\left(z^2\right) \\ H_{2n+1}(z) &= (-1)^n 2^{2n+1} n!\, z L_n^{(1/2)}\left(z^2\right) \end{aligned} \tag{46}$$

As we mentioned in the previous session, these are also very useful when modeling correlated random processes. The elliptic integrals, which we have run into in the first set of exercises, can be accommodated within this scheme as well

$$\begin{aligned} \int_0^{\pi/2} \sqrt{\left(1 - k^2 \sin^2\theta\right)}\,d\theta &= \frac{\pi}{2}\,{}_2F_1\left(1/2, -1/2; 1; k^2\right) \\ \int_0^{\pi/2} \frac{d\theta}{\sqrt{\left(1 - k^2 \sin^2\theta\right)}} &= \frac{\pi}{2}\,{}_2F_1\left(1/2, 1/2; 1; k^2\right) \end{aligned} \tag{47}$$

The Legendre and associated Legendre functions, that arise in the separation of the Laplacian operator in spherical polar co-ordinates, are also expressible in terms of hypergeometric functions:

$$P_n^m(z) = \frac{(-1)^m (n+m)!\left(1-z^2\right)^{m/2}}{2^m (n-m)!\, m!}\, {}_2F_1\left(m-n, n+m+1; m+1; (1-z)/2\right). \tag{48}$$

The list is virtually endless; Abramowitz and Stegun gives a good overview of the useful special functions and their various inter-relations. A great deal can be learnt from these series representations, using relatively simple techniques. (We've mentioned before that *Mathematica* makes extensive use of this approach to special functions.) But the analysis is a bit, well, lack lustre. A great deal of the drudgery involved in special function bashing can be avoided through the introduction of integral representations of the functions. This is what we will do next.

Integral representations of special functions

Rather than deal exclusively with the power series introduced in the previous section, one might try to express the special function as an integral. The usual plan is to take the series expansion, represent part of each term by an integral and interchange the order of summation and integration. If one has been sufficiently cunning the resulting summation can be carried out explicitly, leaving us with an integral that represents the summed up series. There is a lot more you can do with an integral, compared with a series (basically you have all the tricks provided by calculus at your disposal, rather than the rather turgid algebra involved in series manipulation.) A few examples should make this clearer. We start with the hypergeometric function and, at the same time, recall the form taken by Euler's beta function and the binomial theorem:

$${}_2F_1(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} z^n \qquad \int_0^1 t^{a-1}(1-t)^{b-1} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \qquad \frac{1}{(1-x)^{\alpha}} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} x^n \tag{49}$$

When we compare the hypergeometric and binomial series we see that the latter already has one Pochhammer symbol 'upstairs'. Can we use the beta function integral to insert the remaining bits? At a stroke we have

$$\begin{aligned}\int_0^1 dt \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a} &= \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n \int_0^1 dt t^{b+n-1}(1-t)^{c-b-1} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} \frac{\Gamma(n+b)\Gamma(c-b)}{\Gamma(c+n)} z^n \\ &= \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\, {}_2F_1(a,b;c;z)\end{aligned} \tag{50}$$

This is the 'integral representation' we are looking for. If we take the exponential function as a starting point we can build up an integral representation of the confluent hypergeometric function in much the same way:

$${}_1F_1(\alpha;\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n n!} z^n = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 \exp(zt) t^{a-1}(1-t)^{b-a-1} dt \tag{51}$$

To demonstrate the power of this integral representation technique let's change variables $t \to 1-s$ in this; we get

$$\begin{aligned}{}_1F_1(a;b;z) &= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 \exp(z(1-s))(1-s)^{a-1} s^{b-a-1} ds \\ &= \exp(z)\, {}_1F_1(a-b;b;-z)\end{aligned} \tag{52}$$

This result, known as Kummer's transformation, pops out in just a couple of lines. Compare this with the series based derivation:

$$\begin{aligned}\exp(-z)\,{}_1F_1(a;b;z) &= \sum_{p=0}^{\infty}\frac{(-z)^p}{p!}\sum_{q=0}^{\infty}\frac{(a)_q}{(b)_q q!}z^q \\ &= \sum_{r=0}^{\infty}(-z)^r\sum_{q=0}^{r}\frac{(a)_q(-1)^q}{(b)_q q!(r-q)!} \\ &= \sum_{r=0}^{\infty}(-z)^r\frac{1}{r!}\sum_{q=0}^{r}\frac{(a)_q(-r)_q}{(b)_q q!}\end{aligned} \tag{53}$$

To make further progress we must now evaluate $\sum_{q=0}^{r}\frac{(a)_q(-r)_q}{(b)_q q!}$; otherwise we are a bit stuck. For what it's worth this very quantity was evaluated some centuries ago (so long ago in fact that one might be forgiven for forgetting about it); the result is known as Vandermonde's theorem and in our notation can be written as

$$\sum_{q=0}^{r}\frac{(a)_q(-r)_q}{(b)_q q!} = \frac{(b-a)_r}{(b)_r}\ . \tag{54}$$

Substitution of this morsel of arcane knowledge leads us to Kummer's transformation. It may just be a matter of taste, but I think the integral representation approach is much more compact and straightforward. Another example is the evaluation of the hypergeometric series when its argument is unity; the integral representation makes this easy:

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 dt\,t^{b-1}(1-t)^{c-b-a-1} = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-b)\Gamma(c-a)} \tag{55}$$

Vandermonde's theorem is in fact a special case of this result.

We now apply this integral representation approach to derive a result that is potentially very useful (it allows us to simulate clutter with reasonably arbitrary correlation properties, rendering nugatory many years of toil undertaken by past members of the SAR group.) Most of the tricks of the trade are deployed here in a calculation that is relevant to applications, and is worth flogging through as a case study. The thing we want to prove is

$$\sum_{n=0}^{\infty}H_n(x)H_n(y)\frac{\rho^n}{2^n n!} = \left(1-\rho^2\right)^{-1/2}\exp\left(\frac{2xy\rho-\rho^2\left(x^2+y^2\right)}{1-\rho^2}\right) \tag{56}$$

Here the RHS is related to the joint probability density of two correlated Gaussian random variables. If we set $\rho = \exp(-t)$, (56) provides an expansion of the propagator for the Ornstein-Uhlenback process (a route to the simulation of coherent clutter); it is also pertinent to the quantum mechanical description of the harmonic oscillator. The LHS provides an expansion in terms of eigenfunctions of the appropriate adjoint Fokker-Planck or Hamiltonian operator. The integral representation that helps us out is

$$H_n(x)=\frac{2^n}{\sqrt{\pi}}\int_{-\infty}^{\infty}du(x+iu)^n\exp\left(-u^2\right). \tag{57}$$

Before we set about (56), we will verify this representation for even n; the odd order case follows similarly and can be undertaken as an exercise. Thus we have

$$H_{2n}(x)=\frac{2^{2n}}{\sqrt{\pi}}\int_{-\infty}^{\infty}du(x+iu)^{2n}\exp\left(-u^2\right); \tag{58}$$

expanding the integrand through the binomial theorem gives us

$$\begin{aligned}H_{2n}(x)&=\frac{2^{2n}}{\sqrt{\pi}}\int_{-\infty}^{\infty}du(x+iu)^{2n}\exp\left(-u^2\right)\\&=\frac{2^{2n}}{\sqrt{\pi}}\sum_{p=0}^{2n}\frac{(2n)!}{(2n-p)!\,p!}x^{(2n-p)}i^p\int_{-\infty}^{\infty}duu^p\exp\left(-u^2\right)\end{aligned} \tag{59}$$

The odd order p terms vanish, and we can write

$$\begin{aligned}H_{2n}(x)&=\frac{2^{2n+1}}{\sqrt{\pi}}\sum_{p=0}^{n}\frac{(2n)!}{(2n-2p)!(2p)!}x^{(2n-2p)}(-1)^p\int_0^{\infty}duu^{2p}\exp\left(-u^2\right)\\&=\frac{2^{2n}}{\sqrt{\pi}}\sum_{p=0}^{n}\frac{(2n)!}{(2n-2p)!(2p)!}x^{(2n-2p)}(-1)^p\int_0^{\infty}dss^{p-1/2}\exp(-s)\end{aligned} \tag{60}$$

Having made this change of variable to s we recognise the integrals as gamma functions; this gives us

$$H_{2n}(x)=\frac{2^{2n}}{\sqrt{\pi}}\sum_{p=0}^{n}\frac{(2n)!}{(2n-2p)!(2p)!}x^{(2n-2p)}(-1)^p\,\Gamma(p+1/2) \tag{61}$$

Now we develop the coefficient of x through the gamma function duplication formulae

$$(2p)!=\frac{2^{2p}p!\,\Gamma(p+1/2)}{\sqrt{\pi}};\quad (2(n-p))!=\frac{2^{2(n-p)}(n-p)!\,\Gamma(n-p+1/2)}{\sqrt{\pi}} \tag{62}$$

to give

$$\begin{aligned}H_{2n}(x)&=\sqrt{\pi}(2n)!\sum_{p=0}^{n}\frac{x^{(2n-2p)}(-1)^p}{p!(n-p)!\,\Gamma(n-p+1/2)}=\sqrt{\pi}(2n)!(-1)^n\sum_{p=0}^{n}\frac{x^{2p}(-1)^p}{p!(n-p)!\,\Gamma(p+1/2)}\\&=\frac{(2n)!(-1)^n}{n!}\;{}_1F_1\left(-n,1/2,x^2\right)\end{aligned} \tag{63}$$

This ties in with the expression (46) we gave earlier (when we yet again use the gamma function duplication formula). Now we set to to prove the formula; introducing our integral representation we find that

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{\rho^n}{2^n n!} = \frac{1}{\pi}\sum_{n=0}^{\infty} 2^{2n}\int_{-\infty}^{\infty} du(x+iu)^n \exp\left(-u^2\right)\int_{-\infty}^{\infty} dv(y+iv)^n \exp\left(-v^2\right)\frac{\rho^n}{2^n n!} \quad (64)$$

We change the order of summation and integration; the sum is just our old friend the exponential function.

$$\begin{aligned}\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{\rho^n}{2^n n!} &= \frac{1}{\pi}\int_{-\infty}^{\infty} du \exp\left(-u^2\right)\int_{-\infty}^{\infty} dv \exp\left(-v^2\right)\sum_{n=0}^{\infty}\frac{2^n (x+iu)^n (y+iv)^n \rho^n}{n!} \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} du \exp\left(-u^2\right)\int_{-\infty}^{\infty} dv \exp\left(-v^2\right)\exp(2(x+iu)(y+iv)\rho)\end{aligned} \quad (65)$$

What's left boils down to taking the Fourier transform of a Gaussian twice on the trot and so shouldn't present any real difficulties. This is encouraging because the thing we're looking for is itself 'Gaussian'. Filling in the details, we do the v integral first

$$\int_{-\infty}^{\infty} dv \exp\left(-v^2 + 2\rho v(ix-u)\right) = \sqrt{\pi}\exp\left(-\rho^2 x^2\right)\exp\left(\rho^2 u^2 - 2i\rho^2 xu\right) \quad (66)$$

When this is plugged in we find that we have to evaluate

$$\begin{aligned}&\frac{\exp\left(2\rho xy - \rho^2 x^2\right)}{\sqrt{\pi}}\int_{-\infty}^{\infty} du \exp\left(-u^2\left(1-\rho^2\right) + 2i\rho u(y-\rho x)\right) = \\ &\qquad \frac{\exp\left(2\rho xy - \rho^2 x^2 - \rho^2\left(y^2 + \rho^2 x^2 - 2\rho xy\right)/\left(1-\rho^2\right)\right)}{\sqrt{1-\rho^2}}\end{aligned} \quad (67)$$

which boils down to the RHS of (56). So we have derived, and rendered much less mysterious and forbidding, a result that is real practical usefulness. Each of the steps involved has been quite simple; the whole thing is just like a jigsaw puzzle.

We conclude our discussion of integral representations with the Bessel functions. We recall the series representation

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{r=0}^{\infty}\frac{\left(-z^2/4\right)^r}{r!\,\Gamma(\nu+r+1)} \quad (68)$$

How might we construct something like the summand? We note that

$$\int_{-1}^{1} t^{2n+1}\left(1-t^2\right)^{\nu-1/2} dt = 0 \qquad \int_{-1}^{1} t^{2n}\left(1-t^2\right)^{\nu-1/2} dt = \int_{0}^{1} p^{n-1/2}(1-p)^{\nu-1/2} dp = \frac{\Gamma(n+1/2)\Gamma(\nu+1/2)}{\Gamma(\nu+n+1)} \quad (69)$$

From this it is easy to show (using the gamma function duplication formula) that

$$J_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu + 1/2)} \int_{-1}^{1} \exp(izt)\left(1 - t^2\right)^{\nu - 1/2} dt \,. \tag{70}$$

This representation works for all ν, and should be contrasted with the following, which holds only for integer orders.

$$J_n(z) = \frac{i^{-n}}{2\pi} \int_0^{2\pi} \exp(iz\cos\theta)\cos(n\theta)d\theta \tag{71}$$

(you might like to prove this: expand the exponential; represent each power of $\cos\theta$ in terms of cosines of multiples of θ; integrate over θ, exploiting the orthogonality of cosine functions we used to construct Fourier series, tidy up the resulting mess using gamma function identities and re-sum. Again it is just like a jigsaw.) This is the guise in which Bessel functions crop up most regularly in subsequent sessions. A closely related integral representation is that of the modified Bessel function

$$I_n(z) = \frac{1}{2\pi} \int_0^{2\pi} \exp(z\cos\theta)\cos(n\theta)d\theta \tag{72}$$

These appear all over the place when we consider the statistics of a coherent signal in noise, and its enhancement by incoherent averaging. Another Bessel function related integral representation is that of the K distributed intensity

$$P(z) = \frac{b^\nu}{\Gamma(\nu)} \int_0^\infty dx x^{\nu - 2} \exp(-bx - z/x)\,; \tag{73}$$

This embodies the compound representation of non-Gaussian clutter (something we will discuss at length in a few sessions' time) and can be expressed in terms of another sort of modified Bessel function, the so called K function. Such is the power of the integral representation method that you do not need to identify this piece of flash explicitly to work out everything needed to model clutter effectively. You might like to ponder to what extent there is a resemblance between (73) and (71), (72).

This is as far as we need go into background material on special functions. Obviously we have just scratched the surface of the subject, but have probably covered all we will need in future sessions. Should your appetite be whetted you might like to look at some of the following, which provide different perspectives on the properties of a much greater range of functions

M. Abramowitz and I. A. Stegun; 'Handbook of Mathematical Functions', Dover, 1971
An encyclopaedia of formulae, with no proofs etc. and a lot of numerical tables that are now rather redundant.

I. N. Sneddon; 'Special Functions of Mathematical Physics and Chemistry', Oliver and Boyd, 1966
Covers the subject well, using the elementary methods we have looked at today.

E. T. Whittaker and G. N. Watson, 'A Course of Modern Analysis', CUP, 1969.
A hard-core classic, that develops the theory of functions of a complex variable, then applies it magisterially to a host of special functions. Exhaustive and daunting in equal measures; read this and weep.

E.T.Copson, 'Theory of Functions of a Complex Variable', Clarendon Press, 1970
The wimps' Whittaker and Watson. Read this in private and leave W&W casually on your desk.

J. D. Talman, 'Special Functions: a Group Theoretic Approach', Benjamin, 1968
Based on a set of lectures by Wigner, this develops special function theory in the context of representations of Lie Groups. Thus it eschews the theory of functions of a complex variable, appealing to another body of knowledge, that might, or might not, be more familiar.

G. E. Andrews, R. Askey and R. Roy, 'Special Functions', CUP, 2000.
A modern text, which demonstrates that special function theory is alive and well.

Accessing special functions with *Mathematica*

So far we have said very little about how these special functions might be evaluated; we have also had to carry through a lot of rather longwinded and painstaking manipulations. Fortunately both these impediments to the use of special functions can be avoided if one exploits *Mathematica*. To bring this session to a close we will look at *Mma* in action, giving examples of some things that it can do. It is not intended that we learn how to program in *Mma*, or undertake a comprehensive overview of its capabilities; basically we're going to use it like the ultimate pocket calculator. Wolfram's mighty tome - The *Mathematica* Book, as it now calls itself - contains vast amounts of information and can be accessed through the Help command at any time during a *Mma* session. Shaw and Tigg's *Applied Mathematica: Getting Started, Getting it Done*', Addison Wesley, 1994, is a good place to start if you want a 'Copson' rather than a 'Whittaker and Watson' to help you on your way. A *Mma* notebook has been circulated with these notes, which you might like to play with. The rest of the session should be a 'live' run-through of this, that demonstrates *Mathematica*'s way with Algebra, Calculus and Graphics, then does a few special function manipulations. You are urged most fervently to have a go with *Mma*. I believe QinetiQ has a chat room where one can air mathematical issues and that Dr. S. P. Luttrell, who is truly a master of *Mathematica*, keeps an eye on this and is a great help to beginner and seasoned sum-basher alike.

Exercises

The following should provide some useful practice in manipulating special functions. They should be attacked on paper first; *Mathematica* can be used to lessen the load involved in the intermediate steps and as a final check on what you get as an answer. (Remember, however, that *Mathematica* can sometimes get things wrong, so be a bit wary when using it.)

1 Background material on the gamma function (Some of these are hard; if all else fails, just check whether *Mathematica* can 'do' them.)

Prove the gamma function duplication formula as follows:

Show that $\int_0^{\pi/2} \sin^n x \cos^n x dx = \frac{1}{2^n}\int_0^{\pi/2} \sin^n x dx$; evaluate each of these in terms of gamma functions. The duplication formula $\pi^{1/2}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2)$ pops out when you make the identification $z = (n+1)/2$

If you have experience of contour integration, show that $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$, using (22) as your starting point.

Investigate the connection between the integral representation of the gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} \exp(-t) dt$$

and the Gauss product representation

` $$\Gamma(z) = \lim_{n\to\infty} \Gamma(z,n); \quad \Gamma(z,n) = \frac{n!\, n^z}{z(z+1)\cdots(z+n)}$$

To do this show, by integrating by parts, that

$$\Gamma(z,n) = n^z \int_0^1 (1-s)^n s^{z-1} ds = \int_0^n (1-t/n)^n t^{z-1} dt,$$

then let *n* go to infinity.

Use Gauss's product representation to give another proof of the duplication formula.

Verify the following integrals, expressed in terms of the gamma function (these are gratuitously horrid.)

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}} dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)(a+b)^m a^n}$$

$$\int_{-1}^1 \frac{(1+x)^{2m-1}(1-x)^{2n-1}}{(1+x^2)^{m+n}} dx = 2^{m+n-2}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

2 The zeta function has the series representation

$$\varsigma(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

From this, prove the following

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^s} = \varsigma(s)\left(1-2^{-s}\right)$$

$$-\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} = \left(1-2^{1-s}\right)\varsigma(s)$$

$$\varsigma(s) = \frac{1}{\Gamma(s)}\int_0^{\infty} dt \frac{t^{s-1}}{\exp(t)-1} = \frac{1}{\Gamma(s)\left(1-2^{1-s}\right)}\int_0^{\infty} dt \frac{t^{s-1}}{\exp(t)+1}$$

By whatever means you see fit (including a review of the exercises for the previous session) evaluate $\varsigma(2), \varsigma(4), \varsigma(6)$; if you look them up in a book you should also look up the derivation of the answers.

Show that

$$\varsigma(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\prod_p \left(1-1/p^s\right)}$$

Here *n* runs over all positive integers; *p* takes the values of all the *prime* numbers. This establishes a remarkable connection between number theory and analysis. What is the probability that two positive integers, chosen at random, have no common factor?

(The zeta function plays a cameo role in 'A Beautiful Mind'; the lecture given by John Nash, during which it finally becomes clear that he is loosing the plot, is about the zeta function. It is to Russell Crowe's credit that the sums written on the board bear a striking resemblance to this exercise. Such a happy state of affairs should be contrasted with 'Good Will Hunting', in which the sums are absolutely crap.)

3 In our discussion of the Hankel transform in session 4 we mentioned that

$$\int_0^{\infty} dr r J_0(kr)\exp\left(-\alpha r^2\right) = \frac{1}{2\alpha}\exp\left(-k^2/4\alpha\right)$$

Prove this directly then, emboldened by your success, evaluate

$$\int_0^{\infty} dr r^{\beta-1} J_\nu(kr)\exp\left(-\alpha r^2\right)$$

Use *Mathematica* to check your answer.

4 During the session we showed how an even order Hermite polynomial could be expressed as a confluent hypergeometric function (63). Show that an equivalent expression is

$$H_{2n}(x) = (2x)^{2n} {}_2F_0\left(-n, 1/2 - n; -1/x^2\right).$$

Obtain the corresponding results for odd-order Hermite polynomials. Check these results against explicit expressions for the polynomials generated by *Mathematica* (`HermiteH` is the fellow you want). In the previous session we claimed that the Hermite polynomials obey the orthogonality relation

$$\int_{-\infty}^{\infty} \exp\left(-x^2\right) H_n(x) H_m(x) dx = 2^n n! \sqrt{\pi}\, \delta_{m,n} .$$

Use *Mathematica* to verify a few special cases. Prove the general case. To do this it is convenient to show that (58) is equivalent to

$$H_n(x) = (-1)^n \exp\left(x^2\right) \left(\frac{d}{dx}\right)^n \exp\left(-x^2\right)$$

and do a lot of integration by parts. You might like to look into Hermite polynomials a bit more: try an intermediate level quantum mechanics text, Abramowitz and Stegun or Eric's World of Math on the Wolfram website.

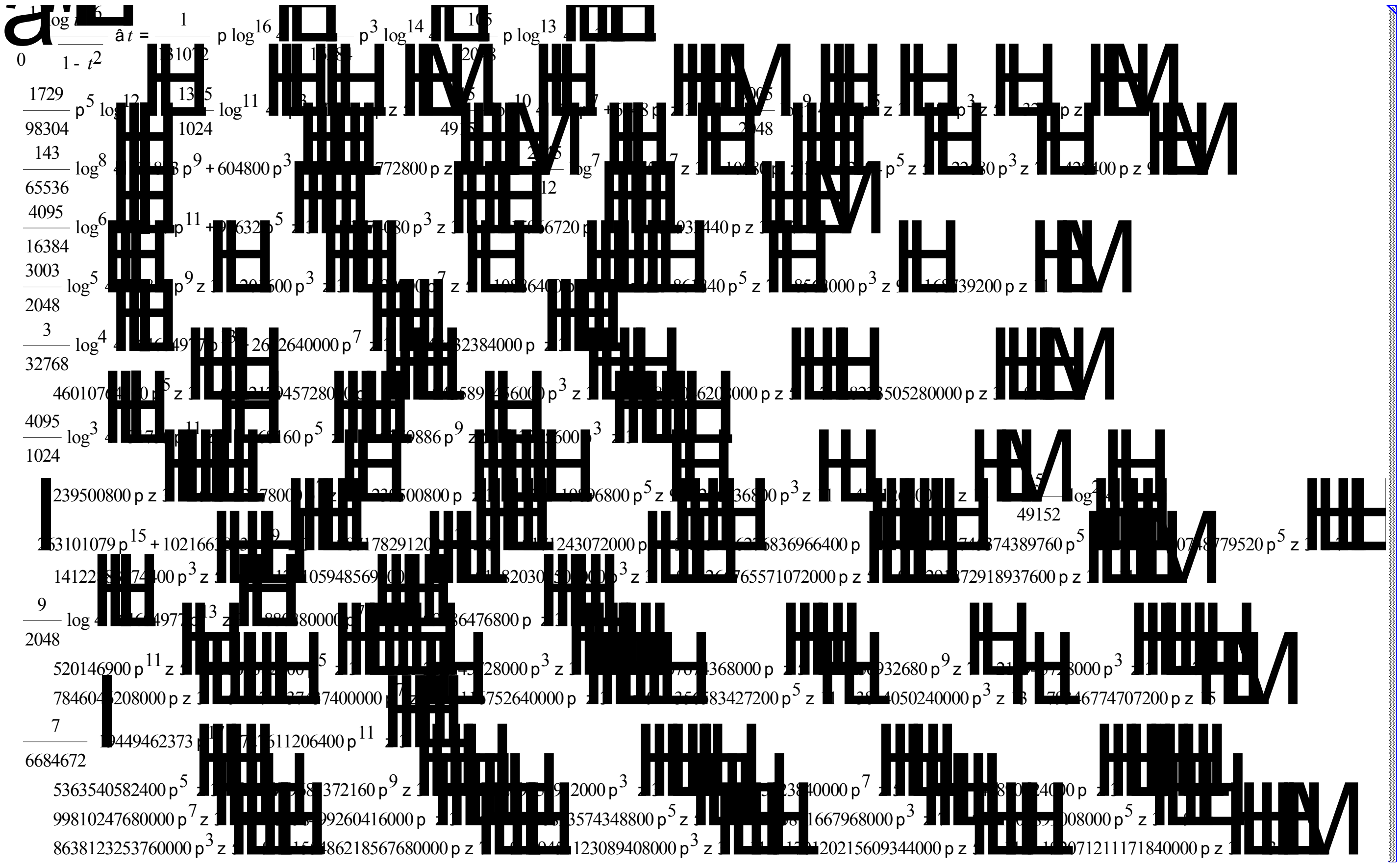
5 In (73) we have the compound form of the K distribution. Use *Mathematica* to evaluate this integral, checking that a *K* does indeed emerge. Consider another integral representation

$$P(z) = \frac{1}{2} \int_0^{\infty} du\, u J_o\left(u\sqrt{z}\right)\left(1 + u^2/4b\right)^{-\nu}$$

Check that this leads to the same form for $P(z)$. Can you demonstrate that this fellow is equivalent to (73) without evaluating either of them explicitly? Evaluate the characteristic function of the K distribution,

$$C(s) = \int_0^{\infty} \exp(-sz) P(z) dz$$

and find out what you can about your answer.



The above, taken from *Mathematica*'s Formula Gallery, is either meant to impress or to provoke a wry smile. Which does it do for you? Some mathematical analysis and comment backing up your position are required.